

## LEPTON FLAVOR NON-CONSERVATION

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### ABSTRACT

In the present work we review the most prominent lepton flavor violating processes ( $\mu \rightarrow e\gamma$ ,  $\mu \rightarrow 3e$ ,  $(\mu, e)$  conversion,  $M - \bar{M}$  oscillations etc), in the context of unified gauge theories. Many currently fashionable extensions of the standard model are considered, such as: *i*) extensions of the fermion sector (right-handed neutrino); *ii*) minimal extensions involving additional Higgs scalars (more than one isodoublets, singly and doubly charged isosinglets, isotriplets with doubly charged members etc.); *iii*) supersymmetric or superstring inspired unified models emphasizing the implications of the renormalization group equations in the leptonic sector. Special attention is given to the experimentally most interesting  $(\mu - e)$  conversion in the presence of nuclei. The relevant nuclear aspects of the amplitudes are discussed in a number of fashionable nuclear models. The main features of the relevant experiments are also discussed, and detailed predictions of the above models are compared to the present experimental limits.

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## 1. INTRODUCTION

All currently known experimental data are consistent with the standard model of weak and electromagnetic interactions (SM). Within the framework of the SM, baryon and lepton quantum numbers are separately conserved. In fact one can associate an additive lepton flavor quantum number with each lepton generation which appears to be conserved. There are thus three such conserved quantum numbers  $L_e$ ,  $L_\mu$  and  $L_\tau$  each one associated with the lepton generations  $(e^-, \nu_e)$ ,  $(\mu^-, \nu_\mu)$ ,  $(\tau^-, \nu_\tau)$ , with their antiparticles having opposite lepton flavor. It is in fact these quantum numbers which distinguish between the three neutrino species if they are massless.

Most theorists, however, view the SM not as the ultimate theory of nature but as a successful low energy approximation. In possible extensions of the SM it is legitimate to ask whether lepton flavor conservation still holds. In fact in such gauge models (Grand Unified Theories, Supersymmetric extensions of the SM, Superstring inspired models) such quantum numbers are associated with global (non local) symmetries and their conservation must be broken at some level.

Motivated in part by this belief the search for lepton flavor violation, which began almost half a century ago (Hincks and Pontecorvo, 1948 [1], Lagarigue and Peyrou, 1952 [2], Lokanathan and Steinberger, 1955 [3], see also Frankel, 1975 [4]) has been revived in recent years and is expected to continue in the near future. In the meantime the number of possible reactions for testing lepton flavor has been increased. The most prominent such reactions are

$$\mu \rightarrow e\gamma \tag{1}$$

$$\tau \rightarrow e\gamma \quad and \quad \tau \rightarrow \mu\gamma \tag{2}$$

$$\mu \rightarrow ee^+e^-, \tag{3}$$

$$\tau \rightarrow ee^+e^-, \quad \tau \rightarrow \mu e^+e^- \tag{4}$$

$$\tau \rightarrow e\mu^+\mu^-, \quad \tau \rightarrow \mu\mu^+\mu^- \quad (5)$$

$$K_L \rightarrow \mu^\pm e^\mp, \quad K^+ \rightarrow \pi^+ \mu e \quad (6)$$

$$(\mu^+e^-) \leftrightarrow (\mu^-e^+) \quad \text{muonium} - \text{antimuonium} \quad \text{oscillations} \quad (7)$$

$$\mu^-(A, Z) \rightarrow e^-(A, Z) \quad (\text{muon} - \text{electron} \quad \text{conversion}) \quad (8)$$

Finally one could have both lepton and lepton flavor violating processes like

$$(A, Z) \rightarrow (A, Z \pm 2) + e^\mp e^\mp \quad (\beta\beta_{ov} - \text{decay}) \quad (9)$$

$$\mu^-(A, Z) \rightarrow e^+(A, Z - 2) \quad (\text{muon} - \text{positron} \quad \text{conversion}) \quad (10)$$

From an experimental point of view the most interesting reactions are (1), (3), (8), (9) and (10). In this report we will only briefly be concerned about the last two reactions.

The problem of lepton flavor non-conservation is connected with the family mixing in the lepton sector. Almost in all models the above process can proceed at the one loop level via the neutrino mixing. However, due to the GIM mechanism in the leptonic sector, the amplitude vanishes in the limit in which the neutrinos are massless. In some special cases the GIM mechanism may not be completely operative even if one considers the part of the amplitude which is independent of the neutrino mass (Langacker and London, 1988 [5], Valle, 1991 [6], Gonzalez-Garcia and Valle, 1992 [7]). Even then, however, the process is suppressed if the neutrinos are degenerate. It should be mentioned that processes (1)-(8) cannot distinguish between Dirac and Majorana neutrinos. Processes (9) and (10) can proceed only if the neutrinos are Majorana particles.

In more elaborate models one may encounter additional mechanisms for lepton flavor violation. In Grand Unified Theories (GUT's) one may have additional Higgs scalars which can serve as intermediate particles at the one or

two loop level leading to processes (1)-(8). In supesymmetric extensions of the standard model one may encounter as intermediate particles the superpartners of the above. Lepton flavor violation can also occur in composite models, e.g. technicolor [8]. In fact, such models have already been ruled out by the present experimental bounds (see next section).

The observation of any of the above processes (eqs. (1)-(10)) will definitely signal new physics beyond the standard model. It will severely restrict most models. It may take, however, even then much more experimental effort to unravel specific mechanisms responsible for lepton flavor violation or fix the parameters of the models. The question of lepton flavor non-conservation has been the subject of several review papers (Scheck, 1978 [9], Costa and Zwirner, 1986 [10], Engfer and Walter, 1986 [11], Vergados, 1986 [12], Melese, 1989 [13], Heusch, 1990 [14], Herczeg, 1992 [15], Schaaf, 1993 [16]). In the present review we will focus our attention on recent theoretical developments of the subject. We will only cover the essential points of the experimental situation since we do not intend to duplicate the recent experimental review which appeared in this journal (Shaaf, 1993 [16]). Furthermore, the reader can find an interesting account of the early experiments by Di Lella [17, 18].

## 2. LEPTON FLAVOR VIOLATING PROCESSES

We have seen in the previous section that lepton flavor violation, if it occurs, can be demonstrated by many reactions (see eqs. (1)-(8)). In this section we are going to examine the most basic features of the experimentally most important processes.

### 2.1. The $\mu \rightarrow e\gamma$ process

As we have already mentioned this is the oldest and perhaps the best studied process. It was expected to proceed quite fast since the muon and electron, with the exception of their mass, are identical and possess identical electromagnetic and weak interactions (for a historical review see Di Lella, 1993 [17], Vergados, 1986 [12]). In such early estimates the branching ratio was (Feinberg, 1958 [19])

$$R = \frac{\Gamma(\mu^+ \rightarrow e^+ \gamma)}{\Gamma(\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu)} \simeq \frac{\alpha}{24\pi} \simeq 10^{-4} \quad (11)$$

which was puzzling since it was already an order of magnitude larger than the experimental limit at that time. We will see, however, in sect. 3.1.1 that we have additional suppression due to the leptonic analogue of the GIM mechanism. In purely left-handed theories the branching ratio takes the form (Vergados, 1986 [12])

$$R = \frac{3\alpha}{32\pi} [\eta_\nu^{(L)} + \eta_N^{(L)}]^2 \quad (12)$$

with

$$\eta_\nu^{(L)} = \sum_j U_{ej}^{(11)} U_{\mu j}^{*(11)} \frac{m_j^2}{m_W^2}, \quad m_j \ll m_W \quad (13)$$

$$\eta_N^{(L)} = \sum_j U_{ej}^{(12)} U_{\mu j}^{*(12)} \left( \frac{m_W^2}{M_j^2} \right) \left[ a \ln \left( \frac{M_j^2}{m_W^2} \right) + b \right], \quad M_j \gg m_W \quad (14)$$

where  $U_{ej}^{(11)}$  ( $U_{\mu j}^{(11)}$ ) is the amplitude for producing the light eigenstate with mass  $m_j \ll m_W$  in the weak interaction of  $e^-$  ( $\mu^-$ ).  $U^{(12)}$  provide the corresponding amplitudes for the heavy neutrino components. For neutrinos less than  $1MeV$  the term  $\eta_\nu^{(L)}$  is negligible. The term  $\eta_N^{(L)}$  also becomes negligible for very heavy neutrinos.

It is clear, therefore, that lepton flavor violating processes are suppressed. Precisely how suppressed one does not know. Thus, experimentalists have not been deterred from pursuing such hard experiments. For purely experimental reasons only positive muons have been considered since, among other things, they do not undergo capture by a nucleus as negative muons do (see sect. 2.2). The experiment consists in the simultaneous detection of a photon and a positron moving essentially back to back with momentum  $p \sim m_\mu c/2$ .

In addition to accidental  $e\gamma$  events, which can be minimized by shielding, the main source of background is radiative muon decay i.e.

$$\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu \gamma \quad (15)$$

in the kinematic regime in which the neutrinos carry away very little energy. In spite of the heroic experimental efforts, this process has not been observed. The best experimental limit

$$R = \frac{\Gamma(\mu^+ \rightarrow e^+ \gamma)}{\Gamma(\mu^+ \rightarrow e^+ \nu_e \bar{\nu}_\mu)} < 4.9 \times 10^{-11} \quad (90\% \text{ CL}) \quad (16)$$

has been set by LAMPF (Bolton *et al.*, 1988 [20]) using the crystal box detector. This is almost an order of magnitude improvement over the previous record,  $R \leq 1.7 \times 10^{-10}$  (Kinnison *et al.*, 1982 [21]). The proposed limit by the new LAMPF detector MEGA (magnetic spectrometer with large solid angle) is

$$R \sim 10^{-12} \quad (17)$$

the final sensitivity depending on the availability of LAMPF beam time (MEGA collaboration [22], Cooper *et al.*, 1985 [23]).

In the same experiment a limit on the branching ratio for two photon emission was indirectly set

$$R \leq 7.2 \times 10^{-11} \quad (90\% \text{ CL}) \quad (18)$$

For comparison we mention the corresponding limits for  $\tau$  decay obtained by the CLEO collaboration (Bean *et al.*, 1993 [24])

$$R = \frac{(\tau \rightarrow \mu \gamma)}{(\tau \rightarrow \text{all})} < 4.2 \times 10^{-6} \quad (90\% \text{ CL}) \quad (19)$$

In the intermediate neutrino mechanism the branching ratio is still given by eq. (12) with the modification  $U_{ej} \rightarrow U_{\tau j}$  provided of course that we ignore all channels other than  $\tau^\pm \rightarrow \mu^\pm \nu_\mu \nu_\tau$  in the denominator (otherwise the  $\tau$ -mass dependence is complicated). Even in this case, however, the branching ratio is expected to be larger crudely speaking by a factor  $(m_{\nu_\tau}/m_{\nu_\mu})^2$  due to the large coupling of the  $\tau$  to the heaviest neutrino mass eigenstate. The branching ratio for  $\tau \rightarrow e \gamma$  is  $R \leq 1.2 \times 10^{-4}$  obtained by Argus (Albercht *et al.*, 1992 [25]). For a complete list see Depommier and Leroy, 1993 [26].



## 2.2. The $\mu \rightarrow ee\bar{e}$ decay

In principle, every model, which allows  $\mu \rightarrow e\gamma$  to proceed, will also allow  $\mu \rightarrow ee^+e^-$ , the only difference being that now the photon can be virtual decaying to an  $e^+e^-$  pair. In such models one expects  $\mu \rightarrow 3e$  to be suppressed by a power of  $\alpha$  i.e.

$$\frac{R(3e)}{R(e\gamma)} \approx \frac{\alpha}{24\pi} \approx 10^{-4} \quad (20)$$

One can construct models, of course, in which the opposite is true (see Bilenky and Petcov, 1987 [27]). Furthermore, since the photon is now virtual, one may have a contribution from the  $E0$  and  $M0$  form factors [12], which vanish at  $q^2 = 0$ , i.e. for real photons. We should emphasize, however, that there exist models which allow  $\mu \rightarrow 3e$  but forbid  $\mu \rightarrow e\gamma$ . Such are e.g. models in which lepton flavor can be mediated by Higgs scalars which are doubly charged [12]. In fact, in such cases  $\mu \rightarrow 3e$  can proceed even at the tree level. In short, if lepton flavor violating mechanisms exist,  $\mu \rightarrow 3e$  has a better chance of being allowed.

In addition to the above theoretical considerations,  $\mu \rightarrow 3e$  offers a number of experimental advantages as well. One can take advantage of the three charged particles appearing in the final state to reduce the background by a variety of timing and kinematic constraints. The chief required qualities for the detector are: good energy resolution, time resolution and precise vertex construction. A detector which meets well these specifications is the SINDRUM at PSI [28]. The best upper limit obtained by such a detector is

$$R = \frac{\Gamma(\mu^+ \rightarrow e^+e^-e^+)}{\Gamma(\mu^+ \rightarrow all)} < 1.0 \times 10^{-12} \quad (90\% \text{ } CL) \quad (21)$$

It is worth noting that, this limit is a two order of magnitude improvement over the limit obtained by the same group (Bertl *et al.*, 1985 [29]), which in turn was an order of magnitude improvement over that of the Dubna group (Korechenko *et al.*, 1976 [30]), which had stood for about 10 years. It is obvious that, such experiments should be encouraged to continue. At least they will provide supplemental information to  $\mu \rightarrow e\gamma$ .

### 2.3. The $(\mu, e)$ conversion in the presence of nuclei

When a negative muon stops in matter, it finally forms a muonic atom captured by a nucleus with a radius of about 200 times smaller than that of the usual atom and a binding energy in the  $KeV$  regime for light and medium nuclei (for heavy nuclei the binding energy is of the order of a few  $MeV$  see below sect. 4.2.1). After it cascades down to the  $1s$  orbit by emitting x-rays, it eventually disappears by decay in flight

$$\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu \quad (22)$$

or by capture by the nucleus

$$\mu^-(A, Z) \rightarrow \nu_\mu(A, Z - 1) \quad (23)$$

The former dominates for light nuclei but it is negligible for heavy nuclei ( $Z > 10$ ). If lepton flavor is not conserved, one can encounter the process

$$\mu^-(A, Z) \rightarrow e^-(A, Z) \quad (\text{muon} - \text{electron} \quad \text{conversion}) \quad (24)$$

The  $(\mu^-, e^-)$  conversion in the presence of nuclei is the most interesting lepton flavor violating process from an experimental point of view. The reason is that, the detection of one particle is sufficient. No coincidence is needed. The energy region of the produced electron is almost background free. There are, however, some sources of background which have to be taken into account. The first is the muon disintegration in orbit mentioned above eq. (22). The electron energy in free muon decay is below  $53MeV$ . The bound muon decay, unfortunately, yields electron in the high energy tail. These electrons can be mistaken with the interesting  $(\mu, e)$  conversion produced electrons due to the finite energy resolution. Good energy resolution is thus a critical and crucial feature of a detector.

The second source of background is radiative muon capture

$$\mu^-(A, Z) \rightarrow \nu_\mu \gamma(A, Z - 1) \quad (25)$$

followed by pair production. This source of background can be eliminated if one considers only  $(\mu, e)$  conversion leading to the ground state of the final

nucleus. Then, by a judicious choice of the target nucleus the maximum electron energy following pair production in (25) can be much smaller than that of the monochromatic electrons of  $(\mu, e)$  conversion [31]. The background free region can be as big as  $2.5MeV$ .

The third source of background, which can be minimized by shielding, comes from electrons produced by cosmic rays. Fortunately, this kind of background can be studied experimentally, since it is easy to accumulate good high statistics data on cosmic rays during the beam off periods.

Another troublesome background comes from radiative pion capture which is followed by pair production. This can lead to copious electrons in the interesting energy region. Fortunately, the life times for muonic atoms are quite long (70ns or longer). So, conversion events can be separated from the prompt background. It is, however, important to eliminate most pion contamination in the beam with electromagnetic separators.

It is clear from the above discussion that, the detectors must have good energy resolution and large solid angle. Such detectors with good detection efficiency must meet conflicting requirements. One usually tries to strike a compromise between the expected sensitivity, which rises with  $Z$ , and the signal to background ratio which drops with  $Z$ . It is also important to ensure that the coherent production exhausts a large fraction of all the conversion electrons. Fortunately this happens to be the case (see sect. 5.2.2).

The first most efficient such detector for  $(\mu, e)$  conversion is the Time Projection Counter (TPC) developed at TRIUMF (Bryman *et al.*, 1993 [32]). Using a target of  $Ti$ , dictated by the above requirements as well as additional experimental constraints, the TRIUMF collaboration has obtained

$$R = \frac{\Gamma(\mu^- Ti \rightarrow e^- Ti)}{\Gamma(\mu \rightarrow all)} < 4.6 \times 10^{-12} \quad (90\% \text{ CL}) \quad (26)$$

(Ahmad *et al.*, 1988 [33]). Using a  $Pb$  target the same group has obtained

$$R = < 4.9 \times 10^{-10} \quad (90\% \text{ CL}) \quad (27)$$

At the same time the experiment found no positron candidate for process (10) with excitation energy in the final nucleus below a few  $MeV$ . This has led to the following limits

$$R = \Gamma(\mu^- Ti \rightarrow e^+ Ca(gs)) < 0.9 \times 10^{-11} \quad (90\% \text{ CL}) \quad (28)$$

The total branching ratio depends, of course, on the final nucleus excitation energy. For a giant resonance distribution with a mean value of  $20MeV$  the authors deduced the limit

$$R = \Gamma(\mu^- Ti \rightarrow e^+ Ca(E < 20MeV)) < 1.7 \times 10^{-10} \quad (90\% \text{ CL}) \quad (29)$$

Another detector with high sensitivity is the SINDRUM II Spectrometer (Badertscher *et al.*, 1991 [34]). With this detector, during the test run, a marginal improvement was achieved for the  $(\mu, e)$  conversion branching ratio

$$R = < 4.4 \times 10^{-12} \quad (90\% \text{ CL}) \quad (30)$$

while the  $(\mu^-, e^+)$  has improved by a factor of 2, i.e.

$$R(\mu^- Ti \rightarrow e^+ Ca(gs)) < 5.5 \times 10^{-12} \quad (90\% \text{ CL}) \quad (31)$$

The next goal set by SINDRUM [34] is

$$R \approx 3 \times 10^{-14} \quad (32)$$

and by MELC (Djilkibaev and Lobashev, 1992 [35]) is

$$R \approx 10^{-16} \quad (33)$$

It is clear from the above discussion that,  $(\mu, e)$  conversion has definite experimental advantages. The proposed limits may meet the predictions of realistic models. Furthermore,  $(\mu, e)$  conversion, like  $\mu \rightarrow 3e$  discussed in the previous section, may occur in a number of models which do not lead to  $(\mu \rightarrow e\gamma)$  such as e.g. those involving the box diagrams (see sect. 3). Due to the fact that, the various mechanisms (discussed in sect. 3) lead to different A and Z dependence,  $(\mu, e)$  conversion, if it is ever observed, may be able to

shed light even on the detailed mechanisms for lepton flavor violation. For this reason it will be discussed in detail in sects. 4 and 5.

## **2.4 Muonium-antimuonium oscillations**

The muonium atom  $M = (\mu^+ e^-)$  has interesting electromagnetic properties compared to positronium. In the presence of lepton flavor changing interactions transition between muonium and antimuonium,  $\bar{M} = (\mu^- e^+)$ , can occur which are the analogue of the well known oscillations in the  $K^0 - \bar{K}^0$  system. The only difference is that the expected oscillation period is much longer compared to the life-time of  $\mu^+$ . Additional complications occur due to the fact that the degeneracy of  $M$  and  $\bar{M}$  may be destroyed in an external magnetic field ( $\geq 0.05G$ ) or due to interactions with matter.

The amplitude for  $M - \bar{M}$  oscillations takes the value of a typical four-fermion weak interaction multiplied by a scale factor  $n_x$  (Vergados, 1986 [12]), which depends on the gauge model, and represents the lepton flavor violating parameter. It is this parameter which we expect to extract from the oscillation measurements. Even though the oscillation time is inversely proportional to  $n_x$ ,

$$\tau \simeq 1.3 \times 10^{-2} n_x^{-1} \quad (34)$$

in the actual experiment one measures the probability for  $\bar{M}$  decay which takes the form

$$P_{\bar{M}} \sim 2.6 \times 10^{-5} n_x^{-2} \quad (35)$$

Further reduction can occur in the presence of a magnetic field, as we have already mentioned above.

An important step towards experimental detection of  $M - \bar{M}$  oscillation was achieved after the development of techniques to produce thermal muonium in vacuum (Marshall *et al.*, 1982, 1988 and Huber *et al.*, 1990 [36]). Another important step was the ability to measure a low energy atomic positron in coincidence with the electron of  $\mu^-$ -decay (Mudinger *et al.*, 1988 [37]) and

the capacity of SINDRUM I (Jungmann *et al.*, 1989 [38]) spectrometer to increase the electron solid angle by roughly a factor of 300.

In spite of the above important steps, the sensitivity of  $M - \bar{M}$  oscillation experiments is limited not only by the smallness of  $n_x$ , but by the additional factor of  $2.6 \times 10^{-5}$  of eq. (35) (Schaaf, 1991 [16]). It may, however, become of interest in some special models, especially those for which the process can occur at tree level (Herczeg and Mohapatra, 1992 [39], Vergados, 1986 [12]).

The highest sensitivity has thus far been achieved by LAMPF (Matthias *et al.*, 1991 [40]). No candidate events were found in the experiment which led to the upper limit for  $P_{\bar{M}}$

$$P_{\bar{M}} < 6.5 \times 10^{-7} \quad (90\% \text{ CL}) \quad (36)$$

This leads to

$$n_x < 0.16 \quad (37)$$

The SINDRUM collaboration plans to reach a sensitivity of  $10^{-10}$  by 1993 with the ultimate of  $10^{-11}$  one year later.

## 2.5. Lepton flavor violating meson decays

At first sight, the best such process seems to be  $\pi^0 \rightarrow \mu^\pm e^\mp$ . It turns out, however, that the branching ratio for this reaction is small, since it has to compete not against a weak competitor but against the electromagnetic decay  $\pi^0 \rightarrow 2\gamma$ . Thus, the most prominent such decays are

$$K_L \rightarrow \mu^\pm e^\mp \quad (38)$$

$$K^+ \rightarrow \pi^+ \mu e \quad (39)$$

in spite of the fact that the experiments have to be done in flight with not so intense beams. Such experimental efforts are expected to intensify, if any of the planned meson factories are ever constructed (TRIUMF, European, Moscow). Historically, such processes have been favored, because of

some kind of prejudice in favor of generation number conservation. In other words, a change of generation in the leptonic sector may be compensated by a corresponding change of generation in the quark sector.

For reasons analogous to those mentioned in the discussion of  $\mu \rightarrow 3e$ , the reaction (39) appears to have some advantages. In other words, it has the advantage of abundant particle identification, which may be used to discriminate against background i.e.  $\pi^+\pi^-$  pairs being mistaken as  $\mu^+\mu^-$  pairs. The best limit ever set for the branching ratio of process (39) is

$$R = (K^+ \rightarrow \pi^+ \mu e) / (K^+ \rightarrow \text{all}) < 2.1 \times 10^{-10} \quad (90\% \text{ CL}) \quad (40)$$

by Lee *et al.*, 1990 [41]. This is almost an order of magnitude improvement over the previous limit  $R < 1.1 \times 10^{-9}$  (BNL E777 experiment, Campagnari *et al.*, 1988 [42]). The branch  $\mu^+e^-$  rather than the branch  $\mu^-e^+$  was selected partly due to the generation argument mentioned above ( $\bar{s} \leftrightarrow \mu^+, d \leftrightarrow e^-$ ) but mainly due to the fact that the positron background is more formidable than the electron background.

At present, the most sensitive limit comes for the reaction (38), i.e.

$$R = \frac{(K_L \rightarrow \mu^+ e^-)}{(K_L \rightarrow \text{all})} < 3.3 \times 10^{-11} \quad (90\% \text{ CL}) \quad (41)$$

which has been set by Arisaka *et al.*, 1993 [43]. Once again, this is an order of magnitude improvement over the previous experimental limit,

$$R < 3 \times 10^{-10} \quad (42)$$

which has been set by the BNL E791 experiment (Cousins *et al.*, 1988 [44]).

### 3. LEPTON FLAVOR VIOLATION IN GAUGE THEORIES

We have pointed out in the introduction that, in the standard model of electroweak and strong interactions neutrinos remain strictly massless and lepton flavor is automatically conserved as a global symmetry of the Lagrangian. In most of the extensions of the standard model, however, there are various sources of lepton flavor mixing and processes like  $\mu \rightarrow e\gamma$ ,  $\mu \rightarrow ee\bar{e}$ ,  $\mu \rightarrow e$

conversion etc., which occur at the one-loop level. In particular, lepton flavor non-conservation appears in the cases where these extensions predict non-zero neutrino masses or an extended higgs sector. Natural candidate theories are Grand Unified Models [45] and Supersymmetric Theories [46]. In this section, we are going to summarize briefly the most important extensions of the standard model. We start with a general discussion of the neutrino mass mechanism when a right handed neutrino is included. We further discuss models with additional scalar particles (singly charged isosinglets, doubly charged isosinglets and isotriplets), and finally we present a brief overview of flavor violating effects in Supersymmetric and String motivated Grand Unified models.

### **3.1. Minimal extensions of the standard model**

**3.1.1. The right handed neutrino.** The most obvious way to extend the standard model is to include the right handed neutrinos. When the right handed neutrino is present, a non-zero Dirac mass term is possible in the theory and a “Kobayashi-Maskawa” leptonic mixing matrix appears, which gives rise to flavor violations.

Moreover, since neutrinos are electrically neutral, they can in principle have Majorana masses, violating the *total* lepton number by two units. In this latter case, new processes may also occur, namely the neutrinoless double beta decay ( $\beta\beta_{0\nu}$ -decay) and muon to positron conversion in the presence of nuclei  $\mu^- + (A, Z) \rightarrow e^+ + (A, Z - 2)$ .

Thus, in the general case one may have the following new mass terms in the Yukawa sector of the theory

$$\mathcal{L}_{mass} = \bar{\nu}_L^0 m \nu_R^{0c} + \bar{N}_L^{0c} m_N N_R^0 + \bar{\nu}_L^0 m_D N_R^0 + \bar{N}_L^{0c} m_D^T \nu_R^{0c} \quad (43)$$

where,  $\nu_L^0 = (\nu_e^0, \nu_\mu^0, \nu_\tau^0)_L$  and  $N_R^0 = (N_e^0, N_\mu^0, N_\tau^0)_R$ , are the left and right handed neutrino weak eigenstates. With

$$\nu_R^{0c} = C(\bar{\nu}_L^0)^T \quad , \quad N_L^{0c} = C(\bar{N}_R^0)^T \quad (44)$$



we denote the conjugate fields, while  $m_\nu, m_D, m_D^T, m_N$  are  $3 \otimes 3$  matrices. Thus, the most general neutrino mass matrix is the  $6 \otimes 6$  matrix

$$\begin{pmatrix} m_\nu & m_D \\ m_D^T & m_N \end{pmatrix} \quad (45)$$

The matrix (45) can be diagonalized by separate left and right unitary transformations. Assuming that the masses of the left handed neutrinos are much lighter than those of the right handed ones, and labeling their eigenstates  $\nu_{jL}$ , and  $N_{jR}$ , respectively, the transformation for the left - handed fields is

$$\begin{pmatrix} \nu_L^0 \\ N_L^{0c} \end{pmatrix} = \begin{pmatrix} S_L^{11} e^{-i\Lambda_\nu} & S_L^{12} e^{-i\Lambda_N} \\ S_L^{21} e^{-i\Lambda_\nu} & S_L^{22} e^{-i\Lambda_N} \end{pmatrix} \begin{pmatrix} \nu \\ N \end{pmatrix}_L \quad (46)$$

For the right-handed fields the transformation is

$$\begin{pmatrix} \nu_R^{0c} \\ N_R^0 \end{pmatrix} = \begin{pmatrix} S_L^{11*} e^{-ia} & S_L^{12*} e^{-i\varphi} \\ S_L^{21*} e^{-ia} & S_L^{22*} e^{-i\varphi} \end{pmatrix} \begin{pmatrix} \nu \\ N \end{pmatrix}_R \quad (47)$$

In eqs. (46) - (47),  $\Lambda_\nu = \Lambda_j(\nu)$  and  $\Lambda_N = \Lambda_j(N)$  are diagonal matrices of arbitrary phases, while  $a = a_j$  and  $\varphi = \varphi_j$  are phases related to the  $\mathcal{CP}$ -eigenvalues of the neutrino eigenstates as

$$(\mathcal{CP})\nu_j(\mathcal{CP})^{-1} = e^{-ia_j}\nu_j, \quad (\mathcal{CP})N_j(\mathcal{CP})^{-1} = e^{-i\varphi}N_j \quad (48)$$

Due to the presence of neutrino mass, the charged leptonic currents remain no longer diagonal. Thus the left handed current becomes [12],

$$j_\mu^L = -2\bar{e}_L^0\gamma_\mu\nu_L^0 + h.c. = -2(\bar{e}_L\gamma_\mu U^{11}\nu_L + \bar{e}_L\gamma_\mu U^{12}N_L) + h.c. \quad (49)$$

with

$$U^{11} = e^{-i\Lambda_e}(S_L^e)^\dagger S_L^{11} e^{-i\Lambda_\nu} \quad (50)$$

$$U^{12} = e^{-i\Lambda_e}(S_L^e)^\dagger S_L^{12} e^{-i\Lambda_N} \quad (51)$$

where  $S_L^e$  is the charged lepton mixing matrix. The charged current involving the conjugate fields becomes

$$(j_\mu^L)^c = -2[\bar{\nu}_R \gamma_\mu e^{ia}(U^{11})^T e_R^c + \bar{N}_R \gamma_\mu e^{i\varphi}(U^{12})^T e_R^c] \quad (52)$$

Thus, the phases  $\alpha_j, \varphi_j$  are in principle measurable. They appear in processes where the conjugate fields are present ( $\beta\beta_{0\nu}$  - decay and  $\mu^- \rightarrow e^+$  conversion). The right-handed current is modified analogously

$$j_\mu^R = 2(\bar{e}_R \gamma_\mu U^{21} \nu_R + \bar{e}_R \gamma_\mu U^{22} N_R + h.c.) \quad (53)$$

where

$$U^{21} = e^{i(\lambda_e + \Lambda_e)} (S_L^e)^T S_L^{21*} e^{-i\alpha} \quad (54)$$

$$U^{22} = e^{i(\lambda_e + \Lambda_e)} (S_L^e)^\dagger S_L^{22*} e^{-i\varphi} \quad (55)$$

Notice that, in the presence of right-handed currents the  $S^{21}$  and  $S^{22}$  matrices are involved. If the right-handed neutrinos are absent, in the above formulae  $U^{12}, U^{21}$  and  $U^{22}$  are zero.  $U^{11}$  is the leptonic analogue of the Kobayashi-Maskawa mixing matrix.

The above considerations apply in most Grand Unified models (GUT's) [45] where non-zero neutrino masses arise naturally. In Supersymmetric GUT's in particular, motivated by the observed merging of the Standard Model gauge coupling constants, there has been a revived interest in determining the low energy parameters of the theory, including the unknown neutrino masses and mixing angles [47], in terms of a few inputs at the GUT scale. The general strategy in these approaches is to use the minimal number of parameters at the GUT scale, so as to have the maximum number of predictions at  $m_W$ . Ultimately, one hopes that this minimal set of parameters at the GUT scale may be justified in terms of a more fundamental theory, such as the String Theory. The advantage of such a procedure is that there are many direct or indirect constraints on the neutrino masses from the rest of fermions. For example, the Dirac neutrino mass  $m_{\nu_D}$  is usually related to the up-quark masses in most of the GUT models. It thus appears challenging to utilize all possible such constraints in the neutrino mass matrix, in order to make definite predictions for the as yet elusive neutrinos, which will then be checked by experiment, this way supporting or excluding such GUT scenarios. In the

next sections we are going to use particular predictive models for fermion masses in our estimations of the flavor violating branching ratios.

**3.1.2. The extended higgs sector.** In this subsection, we are going to review the basic features of minimal extensions of the standard model based on the introduction of new scalar particles [48, 49, 50, 51] which are consistent with the gauge symmetry. Neutrino masses are generated in this case without introducing any right handed neutrino. Possible scalars, which couple in a renormalizable way to leptons, are additional higgs doublet fields  $H^{(k)} = (2, \frac{1}{2})$ ,  $k = 1, 2, \dots$ , a simply charged scalar singlet field  $S^- = (1, -1)$  [48, 49, 50], a doubly charged state  $\Delta^{++} = (1, -2)$  [50] as well as a triplet  $T = (3, -1)$ . All the above states can couple to leptons and create diagrams leading to lepton flavor non-conservation. Their Yukawa couplings are [50]

$$\delta\mathcal{L} = \lambda_{ij}\bar{\ell}_{iL}\ell_{jR}^c S + d_{ij}\bar{\ell}_{iL}\frac{\tau \cdot \mathbf{T}}{\sqrt{2}}\ell_{jR}^c + f_{ij}\bar{e}_{jL}^c e_{iR}\Delta^* + h.c \quad (56)$$

It is possible to conserve lepton number in the above Yukawa Lagrangian, if in the above fields we assign the following lepton numbers:

$$L(S) = 2, \quad L(T) = 2, \quad L(\Delta) = 2 \quad (57)$$

But lepton number can be violated explicitly by cubic as well as quartic terms of the above fields [50]

$$\mu(H^{0*}, -H^{-*}) \begin{pmatrix} H'^+ \\ H'^0 \end{pmatrix} S^- + \mu'(H^{0*}, -H^{-*}) \frac{\tau \cdot \mathbf{T}}{\sqrt{2}} \begin{pmatrix} H'^+ \\ H'^0 \end{pmatrix} + \dots \quad (58)$$

where  $H'$  is a second doublet and  $\mu, \mu', \dots$  are the cubic coupling mass parameters. (Notice that  $S^-$  couples antisymmetrically to isospin 1/2 particles.)

A generalization of the above model involves, one additional higgs doublet  $H'$ , the additional isosinglet fields  $S^{ab}$  and the singlet scalars  $\Phi^{ab}$ , where  $a, b$  stand for  $e, \mu, \tau$ . Here, rather than breaking the lepton flavor explicitly [48, 49, 50], we prefer to follow the approach of references [52, 53] and consider models that above some scale  $V$ , have a global abelian lepton flavor symmetry  $G = U(1)_e \times U(1)_\mu \times U(1)_\tau$ . At the scale  $V$ , the group  $G$  is broken spontaneously

Figure 1: One loop contribution to the neutrino masses in the model with an extended Higgs sector.

by vacuum expectation values (vev's) of the singlet scalars  $\Phi^{ab}$  giving rise to Goldstone bosons:  $\Phi^{e\mu} = e^{iF_{e\mu}/V_{e\mu}}$ ,  $\Phi^{e\tau} = e^{iF_{e\tau}/V_{e\tau}}$  and  $\Phi^{\mu\tau} = e^{iF_{\mu\tau}/V_{\mu\tau}}$ . Thus, the new interactions in the Lagrangian are [52, 53]

$$g'_a l_a e^{ca} H' + \lambda_{ab} l_a l_b S^{ab} + \tilde{g}_{ab} H H' S^{ab} \Phi_{ab}^*. \quad (59)$$

Neutrino masses are generated by the one-loop diagram shown in fig. 1. The loop calculation for this graph gives:

$$m_{ab} = \frac{1}{16\pi^2} \lambda_{ab} \tilde{g}_{ba} < \Phi^{ba} > \frac{v(g'_a m_a + g'_b m_b)}{M_S^2 - m_W^2} \ln \frac{M_S^2}{m_W^2} \quad (60)$$

In the above,  $m_a$ ,  $m_b$  are the masses of the charged leptons,  $M_S$  is the mass of the heavy charged singlet  $S^{ab}$  which appears in the loop, while  $v$  is the vev of the standard higgs doublet. We have also assumed that, the second doublet circulating in the loop, which does not develop a  $vev$ , has a mass of order  $m_W$ .

In the most general case, where all the singlet fields  $\Phi^{ab}$  acquire vev's, the neutrino mass matrix can be parametrized as follows [53]:

$$m_\nu = m_0 \begin{pmatrix} 0 & \tan\theta & \cos\phi \\ \tan\theta & 0 & \sin\phi \\ \cos\phi & \sin\phi & 0 \end{pmatrix} \quad (61)$$

In the above matrix  $m_0$  sets the mass scale and is given in terms of the various parameters entering eq. (60) from the formula

$$m_0 = \frac{1}{16\pi^2} \frac{v(V_{e\tau}^2 + V_{\mu\tau}^2)^{\frac{1}{2}}}{M^2 - m_W^2} m_\tau \ln \frac{M^2}{m_W^2} \quad (62)$$

where we have made the approximations  $m_\tau + m_\mu \approx m_\tau$  and  $m_\mu + m_e \approx m_\mu$ , while

$$V_{ab} = \lambda_{ab} \tilde{g}_{ba} g'_a < \Phi^{ba} > \quad (63)$$

Furthermore, since we know nothing about the couplings  $g'_a$ , in the last equation we have used the approximation  $g'_e \approx g'_\mu \approx g'_\tau \approx g'$ . Finally, we have defined

$$\tan\phi = \frac{V_{\mu\tau}}{V_{e\tau}} \quad (64)$$

and

$$\tan\theta = \frac{m_\mu V_{e\mu}}{m_\tau V_{\mu\tau}} \sin\phi \quad (65)$$

In order to diagonalize the above matrix, guided by phenomenological reasons, we make the natural assumption that  $\sin 2\phi \ll 1$ . In this case, the eigenmasses are found to be

$$m_{\nu_1} \approx -\frac{1}{2} m_0 \sin 2\theta \sin 2\phi \quad (66)$$

$$m_{\nu_2} \approx -\frac{m_0}{\cos\theta} + \frac{1}{4} m_0 \sin 2\theta \sin 2\phi \quad (67)$$

$$m_{\nu_3} \approx \frac{m_0}{\cos\theta} + \frac{1}{4} m_0 \sin 2\theta \sin 2\phi \quad (68)$$

Furthermore, we get two different sets of eigenstates, depending upon whether  $\sin\phi \ll 1$  or  $\cos\phi \ll 1$ . Thus, in the case where  $\cos\phi \ll 1$ , the eigenstates are related to the weak states as follows [53]:

$$\begin{aligned} \nu_e &\approx \cos\theta (\sin\phi - \cos^2\theta \cos\phi \sin 2\phi) \nu_1 \\ &+ \frac{1}{\sqrt{2}} (\sin\phi \sin\theta - \cos\phi) \nu_2 + \frac{1}{\sqrt{2}} (\sin\phi \sin\theta + \cos\phi) \nu_3 \\ \nu_\mu &\approx \cos\theta (\cos\phi - \cos^2\theta \sin\phi \sin 2\phi) \nu_1 \\ &+ \frac{1}{\sqrt{2}} (\cos\phi \sin\theta - \sin\phi) \nu_2 + \frac{1}{\sqrt{2}} (\cos\phi \sin\theta + \sin\phi) \nu_3 \end{aligned}$$

$$\begin{aligned}
\nu_\tau &\approx -\sin\theta\nu_1 + \frac{1}{\sqrt{2}}\cos\theta(1 - \sin 2\phi \sin\theta)\nu_2 \\
&+ \frac{1}{\sqrt{2}}\cos\theta(1 + \sin\theta \sin 2\phi)\nu_3
\end{aligned} \tag{69}$$

A simplified version of the above model arises when one considers the particular value  $\cos\phi = 0$ . This case corresponds to the particular choice  $\langle \Phi_{e\tau} \rangle = 0$ , which exhibits an exact  $L_{e-\mu+\tau}$  symmetry. In this latter case, one finds a zero mass for the first neutrino and two completely degenerate states for the other two, i.e. [52, 53]

$$m_{\nu_1} = 0, \quad m_{\nu_2} = -\frac{m_0}{\cos\theta}, \quad m_{\nu_3} = \frac{m_0}{\cos\theta}$$

The diagonalizing matrix is given by

$$U = \begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\sin\theta & \frac{\cos\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} \end{pmatrix} \tag{70}$$

In the last two equations there are only two parameters, namely, the mass scale  $m_0$  and the angle  $\theta$ . If we adopt the MSW-solution for the solar neutrino problem, and a natural hierarchy between the  $m_{\nu_1}$  and  $m_{\nu_2}$  neutrino mass eigenstates, both parameters can be fixed uniquely. (We mention here that an alternative solution to the solar neutrino “puzzle” is based on the assumption of a large neutrino magnetic moment; for a recent review see Pulido [54].)

The model under consideration gives the following formulae for the various oscillation probabilities

$$P(\nu_e \leftrightarrow \nu_\tau) \approx \frac{1}{2}\sin^2 2\theta \left(1 - \frac{1}{2}\sin^2 \pi \frac{L}{l_{23}}\right)$$

$$P(\nu_\tau \rightarrow \nu_\mu) \approx \cos^2 \theta \sin^2 \pi \frac{L}{l_{23}}$$

$$P(\nu_e \rightarrow \nu_\mu) \approx \sin^2 \theta \sin^2 \pi \frac{L}{l_{23}}$$

Here, the short oscillations have been averaged out. We notice that, the above oscillation probabilities are expressed solely in terms of two parameters,  $\theta$  and  $l_{23}$ . If the MSW-effect is interpreted through  $\nu_e \leftrightarrow \nu_\mu$  oscillations, then, from the experimental data [55] one finds that

$$m_{\nu_2} \approx (1.79 - 3.46) \times 10^{-3} eV, \quad \sin\theta \approx (0.71 - 1.10) \times 10^{-1} \quad (71)$$

(see also [56]). We notice, however, that the above model can also accommodate a relatively “large” neutrino mass without creating any particular problem in low energy phenomenology. As an example, we mention the solution given by the proposed model [52, 53] to the  $17KeV$  neutrino “puzzle”.

Other neutrino mass mechanisms have also been proposed in the context of Grand Unified Theories. Of particular interest is the Witten mechanism [57], which is possible [58] in all GUT-models which include the right hand neutrino in a larger representation. The advantage of this mechanism lies in the fact that the Higgses need not belong to large representations of the corresponding symmetry. Other mechanisms giving mass at the two loop level have also been discussed in models predicting large magnetic moments [59].

### **3.2. The flavor violating decays**

We are now in a position to discuss the various violating processes and obtain numerical expressions for the corresponding branching ratios. The processes under consideration are the flavor violating processes  $\mu \rightarrow 3\gamma$ ,  $\mu \rightarrow 3e$ ,  $(\mu^-, e^-)$  conversion as well as the lepton number violating processes  $(\mu^-, e^+)$  and  $\beta\beta_{\nu\nu}$ -decay.

**3.2.1. The  $\mu \rightarrow e\gamma$  and  $\mu \rightarrow 3e$  decay rates.** We start with the familiar  $\mu \rightarrow e\gamma$  decay. From the theoretical point of view, this decay, due to its importance, has been discussed extensively in the literature the last two decades [60]. The essential features and experimental limits of this process have been discussed in sect. 2.1. This decay violates lepton flavor and can

proceed through diagrams which involve massive neutrinos or new scalars predicted in various extensions of the standard theory.

The most general form for the on-shell ( $q^2 = 0$ ) amplitude for  $\mu \rightarrow e\gamma$  is given by

$$\mathcal{M}(\mu \rightarrow e\gamma) = \bar{e}(f_{E1} + f_{M1}\gamma_5)\not{e}m_\mu\sigma_{\rho\nu}q^\rho\epsilon^\nu\mu \quad (72)$$

where  $\epsilon^\nu$  is the photon polarization vector and  $\sigma_{\rho\nu} = \frac{i}{2}[\gamma_\rho, \gamma_\nu]$ . Once the form factors  $f_{E1}$ ,  $f_{M1}$  are given, in a certain gauge model one can easily obtain the branching ratio with respect to  $\mu \rightarrow e + \bar{\nu}_e + \nu_\mu$  decay i.e.

$$R(\mu \rightarrow e\gamma) = 24\pi^2(2\pi\alpha) \frac{|f_{E1}|^2 + |f_{M1}|^2}{G_F^2 m_\mu^4} \quad (73)$$

In the mass mechanism there are basically four diagrams contributing to the amplitude [12]. We distinguish the following cases:

*i) Left-handed currents only:*

The form factors in this case are

$$|f_{E1}| = |f_{M1}| = \frac{G_F}{\sqrt{2}} \frac{m_\mu^2}{32\pi^2} (\eta_\nu^{LL} + \eta_N^{LL}) \quad (74)$$

where  $\eta_\nu^{LL}$ ,  $\eta_N^{LL}$  are lepton violating parameters defined for light and heavy neutrinos, respectively, as

$$\eta_\nu^{LL} = \sum U_{ei}^{(11)} U_{\mu i}^{(11)*} \frac{m_i^2}{m_W^2} \quad (75)$$

$$\eta_N^{LL} = \sum U_{ei}^{(12)} U_{\mu i}^{(12)*} \frac{m_W^2}{M_i^2} \left( a \ln \frac{M_i^2}{m_W^2} + b \right) \quad (76)$$

with  $a = 2$  and  $b = -3$ . The above parameters in eq. (76) include the mixing which is model dependent. In the case of a particular mass matrix ansatz for the  $SO(10)$  model, given for example in ref. [61], we have

$$\eta_\nu^{LL} \approx 5 \times 10^{-20}, \quad \eta_N^{LL} \approx 0 \quad (77)$$

which lead to a suppressed branching ratio



$$R \sim 10^{-46} \quad (78)$$

Thus in left-handed theories  $\mu \rightarrow e\gamma$  is unobservable.

*ii) Right-handed currents (R-R couplings):*

In this case, we have further suppression due to the presence of the intermediate boson  $W_R$ , which is assumed to be much heavier than its left-handed partner

$$\left(\frac{m_{W_L}}{m_{W_R}}\right)^2 = k \leq \frac{1}{10} \quad (79)$$

Assuming that the mixing  $\zeta$  of the two bosons  $W_L$ ,  $W_R$  is small enough ( $\zeta \sim 0.1k$ ), one obtains the following lepton violating parameters

$$\eta_\nu^{RR} = k^2 \sum U_{ei}^{(21)} U_{\mu i}^{*(21)} \frac{m_i^2}{m_W^2} \quad (80)$$

$$\eta_N^{RR} = k^2 \sum U_{ei}^{(22)} U_{\mu i}^{(22)*} \frac{m_W^2}{M_i^2} \left( a \ln \frac{M_i^2}{m_W^2} + b \right) \quad (81)$$

which again lead to an unobservable branching ratio  $R \leq 10^{-36}$  depending on the precise value of  $k$ .

*iii) Left - Right mixing:*

This is the most favorable case in the mass mechanism, since the obtained amplitude has different structure from the previous ones. We get

$$f_{E1,M1}^{LR(RL)} = 6 \frac{G_F}{\sqrt{2}} \frac{m_\mu^2}{32\pi^2} \left\{ \eta_\nu^{LR(RL)} + \eta_N^{LR(RL)} \right\} \quad (82)$$

where  $(\eta^{LR(RL)})_{\nu,N}$  refer to the L(R)-handed coupling for the  $\mu\nu W(e\nu W)$  vertex and to the R(L)-handed coupling for the  $e\nu W(\mu\nu W)$  vertex. Their expressions are given by

$$\eta_\nu^{LR} = \zeta \sum U_{ei}^{(21)} U_{\mu i}^{(11)*} \frac{m_i}{m_\mu} \quad (83)$$

Figure 2:  $\mu \rightarrow e\gamma$  via the singlet field  $S$ .

$$\eta_\nu^{RL} = \zeta \sum U_{ei}^{(11)} U_{\mu i}^{(21)*} \frac{m_i}{m_\mu} \quad (84)$$

$$\eta_N^{LR} = \zeta \sum U_{ei}^{(12)} U_{\mu i}^{(22)*} \frac{m_W^2}{m_\mu M_i} \left( a \ln \frac{M_i^2}{m_W^2} + b \right) \quad (85)$$

$$\eta_N^{RL} = \zeta \sum U_{ei}^{(22)} U_{\mu i}^{(12)*} \frac{m_W^2}{m_\mu M_i} \left( a \ln \frac{M_i^2}{m_W^2} + b \right) \quad (86)$$

However, even in this most favourable case, in the mass mechanism the branching ratio turns out to be very small:

$$R^{L,R} \sim R_N^{L,R} \simeq 10^{-32} \quad (87)$$

From the above discussion it is clear that, the neutrino mass mechanism is in all cases negligibly small due to the tiny mixing and since  $m_j \ll m_W$  or  $M_j \gg m_W$ .

*iv) extended higgs sector:*

It is possible to avoid the suppression mass mechanism by using an expanded Higgs sector. In sect. 3.2, we discussed such a possibility by introducing the isosinglet  $S^-$ . With the diagrams given in fig. 2 one can compute the branching ratio for the  $\mu \rightarrow e\gamma$  decay [50], which is given in terms of the mass  $M_S$  of the isosinglet. One finds [62]

$$R_S = \frac{\alpha}{48\pi} \frac{1}{G_F^2} \left( \frac{\lambda_{\mu\tau} \lambda_{\tau e}}{M_S^2} \right)^2 \leq 4.9 \times 10^{-11} \quad (88)$$

which results to a bound for the singlet mass  $M_S$

$$M_S \geq 94 \text{ GeV} \times [10^2 (\lambda_{\mu\tau} \lambda_{\tau e})^{1/2}] \quad (89)$$

- Another interesting process, which can occur with more mechanisms than  $\mu \rightarrow e\gamma$ , is the process  $\mu \rightarrow \bar{e}ee$ . Firstly, it can occur in all mechanisms which

allow  $\mu \rightarrow e\gamma$  decay, provided that the emitted photon is virtual. Secondly, this process can be mediated by the neutral boson  $Z$ , which finally decays to an  $e^+e^-$  pair. Finally, we can include now box diagrams [12]. Similar diagrams are also generated in the case of the Higgs singlet  $S$ .

In the mass mechanism we can have contributions from all classes of diagrams. We discuss here the most important ones.

1). For purely left handed theories by comparing  $\mu \rightarrow \bar{e}ee$  branching ratio with the corresponding one for  $\mu \rightarrow e\gamma$  decay, one obtains [12]

$$R\left(\frac{\mu \rightarrow 3e}{\mu \rightarrow e\gamma}\right) \approx 1.4 \times 10^{-2} \quad (90)$$

The  $Z$  and  $W$  diagrams give a larger contribution than that of the photonic ones in the case of light neutrinos  $R_Z(\mu \rightarrow 3e) \sim 9 \times 10^2 R_\gamma(\mu \rightarrow 3e)$ ,  $R_W(\mu \rightarrow 3e) \sim 1.8 \times 10^3 R_\gamma(\mu \rightarrow 3e)$ , but still far from the experimental limit.

2). With the presence of the right handed currents, the only important contribution comes from the left - right mixing in the photonic diagrams. We get

$$R \sim 2.7 \times 10^{-5} |\eta_\gamma^{RL}|^2 \quad (91)$$

with

$$|\eta_\gamma^{RL}| = |\eta^{LR} + \eta^{RL}|^2 + |\eta^{LR} - \eta^{RL}|^2 \quad (92)$$

where

$$\eta^{LR(RL)} = \eta_\nu^{LR(RL)} + \eta_N^{LR(RL)} \quad (93)$$

Using the values of the previous sections for the neutrino masses and mixings one obtains

$$R(\mu \rightarrow 3e) \leq 10^{-35} \quad (94)$$

3). We can also introduce here the singlet  $S^-$  to avoid the mass mechanism suppression. Again three classes of diagrams are generated [12]. The most important contribution comes from the photonic ones. The decay rate is given by

$$\Gamma_S(\mu \rightarrow 3e) = \frac{25}{2} m_\mu^3 \frac{\alpha^3}{\pi} |f_{M1}|^2 \quad (95)$$

and the branching ratio by

$$R_S(\mu \rightarrow 3e) \sim 0.73 \times 10^{-2} R_S(\mu \rightarrow e\gamma) \quad (96)$$

**3.2.2.  $(\mu^- - e^-)$  conversion decay rates.** The next flavor changing process we are going to consider is the  $(\mu^-, e^-)$  conversion in the presence of nuclei. The diagrams contributing to this process are similar to those we have considered for  $\mu \rightarrow \bar{e}ee$  decay (see ref. [12]).

Starting again with the photon diagrams, we write down the decay rate which is

$$\Gamma = 8m_\mu^3 \frac{Z_{eff}^4}{Z} \alpha^5 \frac{E_e p_e}{m_\mu^2} \frac{1}{8\pi} \xi_0^2 |ME|^2 \quad (97)$$

where  $\xi_0^2 = |f_{E0} + f_{M1}|^2 + |f_{M0} + f_{E1}|^2$  and  $|ME|$  is the nuclear matrix element. (A detailed discussion will be presented in sects. 4 and 5.) For a wide range of nuclei and assuming only ground state transitions the matrix element  $|ME| \rightarrow M_{gs \rightarrow gs}$  lies in the range

$$|ME| \rightarrow M_{gs \rightarrow gs} \simeq (0.2 - 0.9) Z^2 \quad (98)$$

In the rest of this section we assume an average value  $|ME| \approx 0.5 Z^2$ . The branching ratio is obtained with respect to ordinary muon capture  $\mu^- + (A, Z) \rightarrow \nu_\mu + (A, Z - 1)$  decay rate which is [63, 64]

$$\Gamma(\mu \rightarrow \nu_\mu) = m_\mu^5 \frac{\alpha^3}{2\pi} G_F^2 Z_{eff}^4 Z [F_V^2 + 3F_A^2 + F_P^2 - 2F_A F_P] f(A, Z) \quad (99)$$

where  $F_V, F_A, F_P$  are form factors and the Primakoff's function  $f(A, Z)$  for nuclei with  $A \approx 2Z$  has the value  $f(A \approx 2Z, Z) \approx 0.16$ . Thus

$$R_\gamma \left( \frac{\mu^- \rightarrow e^-}{\mu \rightarrow \nu_\mu} \right) \sim 10^{-3} \alpha^2 Z \frac{E_e p_e}{m_\mu^2} |n_\gamma|^2 \quad (100)$$

where  $n_\gamma$  is related to the lepton violating parameters already defined in previous processes. It is obvious that, even in the L-R mixing where  $|n_\gamma|^2$  gets its maximum value  $\sim 9 \times 10^{-26}$ , the contribution remains small. Thus, for light neutrinos

$$R_\gamma(\mu^- \rightarrow e^-) \sim 10^{-33} Z \frac{E_e p_e}{m_\mu^2} \quad (101)$$

For the  $Z$ -diagrams the basic contribution arises in the L-L case. We thus get the branching ratio

$$R_Z(\mu^- \rightarrow e^-) \approx \frac{g^2}{32\pi^2} \frac{E_e p_e}{m_\mu^2} \frac{A^2}{8Z} \frac{|F_{ch}(q^2)|^2}{f(A, Z)} |n_Z^L|^2 \approx 0.34 \times 10^{-2} |n_Z^L|^2 Z^2 \frac{E_e p_e}{m_\mu^2} \quad (102)$$

where

$$\eta_Z^L \approx \left( \frac{3}{2} + \ln \frac{\langle m_q^2 \rangle}{m_W^2} \right) \eta_\nu^{LL} + \eta_N^{LL} \approx 4 \times 10^{-23} \quad (103)$$

The quantity  $\langle m_q^2 \rangle$  represents the effective quark mass. In the case of box diagrams, the L-L currents are also the most important and the branching ratio is found to be

$$R_W(\mu^- \rightarrow e^-) \approx 0.6 \alpha^2 Z \frac{E_e p_e}{m_\mu^2} (\eta_W^\kappa - 4\eta_W^a)^2 \quad (104)$$

where

$$\eta_W^{\kappa(a)} = \left( 1 + \ln \frac{\langle m_q^2 \rangle}{m_W^2} \right) \eta_\nu^{LL} + \eta_N^{LL} \quad (105)$$

Here  $\kappa$  ( $a$ ) stand for the down (up) quarks. If we ignore the mixing in the quark sector and take  $m_u \sim m_d \sim 4 \times 10^{-2} GeV$ , we get  $R_W \sim 10^{-24} Z E_e p_e / m_\mu^2$

for light neutrinos, which is better than the previous cases. The presence of the right-handed currents in the box diagrams again appear to give negligible contribution.

Although  $\mu - e$  conversion turns out to be negligible in the above context, as will be discussed in the following sections, one finds that this process is enhanced in supersymmetric GUTs, when renormalization group corrections are taken into account. Let us, however, mention here that, in theories with extended fermion sector and a new neutral gauge boson  $Z'$ , one may also have a significant impact in the  $\mu - e$  conversion. It is argued [65] that in such theories, the present experimental limits on the above process give a nuclear model independent bound on the  $Z - e - \mu$  vertex, which is twice as strong as that obtained from  $\mu \rightarrow eee$  decay. In particular, in the case of the  $E_6$  models [66], these limits provide stringent constraints [65] on the mass ( $M_Z \geq 5TeV$ ) and the mixing angle of  $Z$  and  $Z'$ .

**3.2.3.  $\beta\beta_{0\nu}$  and  $(\mu^- - e^+)$  processes.** We discuss now briefly the lepton number violating processes eqs. (9), (10) (see also ref. [12, 67]).

- The oldest lepton violating process, intimately related to the nature of the neutrino, is the neutrinoless  $\beta\beta_{0\nu}$  decay

$$(A, Z) \rightarrow (A, Z \pm 2) + e^\mp + e^\mp, \quad e_b^- + (A, Z) \rightarrow (A, Z - 2) + e^+ \quad (106)$$

which together with the allowed  $\beta\beta_{2\nu}$  decay

$$(A, Z) \rightarrow (A, Z \pm 2) + e^\mp + \left\{ \begin{array}{l} 2\bar{\nu}_e \\ 2\nu_e \end{array} \right\}, \quad e_b^- + (A, Z) \rightarrow (A, Z - 2) + e^+ + \nu_e + \bar{\nu}_e \quad (107)$$

are the only modes of some otherwise absolutely stable nuclei. One finds that [12], the life time of these processes is given by

$$T_{1/2}(0\nu) = K_{0\nu}/|n|^2|ME|_{0\nu}^2 \quad (108)$$

$$T_{1/2}(2\nu) = K_{2\nu}/|ME|_{2\nu}^2 \quad (109)$$

where  $\eta$  is the relevant lepton violating parameter and  $|ME|$  the corresponding nuclear matrix elements. The quantities  $K_{0\nu}, K_{2\nu}$  are functions of  $(A, Z)$  which can take the following values [12]

$$1.5 \times 10^{13}y < K_{0\nu} < 2.5 \times 10^{17}y, \quad (110)$$

$$2.5 \times 10^{19}y < K_{2\nu} < 3.3 \times 10^{26}y \quad (111)$$

Thus,  $\eta$ , which contains all information about the gauge models, should not be much smaller than  $10^{-6}$  (the present experimental limit) in order  $\beta\beta_{0\nu}$  decay to be within the capabilities of present experiments. In the  $SO(10)$  model [61] discussed above, the best value for  $\eta$  is given in the case of light neutrinos  $\eta_\nu^{LL} \sim 4 \times 10^{-9}$ . It is not difficult, however, to invent scenarios where  $\beta\beta_{0\nu}$  decay is not far from the experimental capability of the near future experiments.

In particular, there have been recently discussed cases [68] with three light neutrinos predicting  $\beta\beta_{0\nu}$  decay half-life in the range of the sensitivity of future experiments, induced by the exchange of majorana neutrinos with an effective neutrino mass  $\sim (0.1 - 1.0)eV$ . The interesting feature of these scenarios is that they also provide a solution to the solar neutrino problem and in some particular cases, they can also accommodate a solution to the atmospheric neutrino problem.

- The other interesting lepton number violating process is the  $(\mu^-, e^+)$  conversion

$$\mu_b^- + (A, Z) \rightarrow e^+ + (A, Z - 2)^* \quad (112)$$

The half-life time is given by

$$T_{1/2}(\mu^-, e^+) = K'_{0\nu}/|n'|^2|ME'_{0\nu}|^2 \quad (113)$$

where,  $K'_{0\nu}(\mu^-, e^+) = 2.2 \times 10^{10}yA^{2/3}/(Z_{eff}^4/Z)$  or  $5 \times 10^7y < K'_{0\nu} < 6 \times 10^8y$ , for nuclei ranging from  $^{58}Ni$  to  $^{12}C$  [67]. Even though this process is  $10^{10}$  faster than its sister  $(e^-, e^+)$ , unfortunately it must compete against ordinary muon capture  $\mu^- \rightarrow \nu_\mu$ . Thus, one gets the branching ratio

$$R = \frac{\Gamma(\mu^-, e^+)}{\Gamma(\mu^-, \nu_\mu)} \approx 1.5 \times 10^{-21} \frac{|\eta'| |ME'|^2}{Z(1.62Z/A - 0.62)} \quad (114)$$

where  $\eta'$  is the corresponding lepton violating parameter. Thus, for  $|\eta'| \approx |\eta| \leq 10^{-6}$ , this process is unobservable. Even if transitions to all final nuclear states are considered [67],  $|ME'|^2 \approx 0.1Z^2 = 100$ . The present experimental limit is  $R < 9 \times 10^{-12}$  (see sect. 2.3).

**3.2.4. Other lepton flavor violation mechanisms.** We mention here some other mechanisms of lepton flavor violation which are also possible. Of particular interest is the Bjorken-Weinberg [69] mechanism, based on a two loop contribution when additional Higgs bosons are present. The result is sensitive to the various unknown parameters (Higgs mass etc. [70]), but it can in principle fall close to the experimental limit.

Another possibility of lepton flavor violation arises, if we extend the scalar part of the theory with the introduction of a doubly charged singlet  $\Delta^{--}$  already mentioned in the sect. 3.1.2. The most important effect of this scalar is in the  $\mu \rightarrow 3e$  decay, which is mediated at the tree level [50]. One finds that the branching ratio is

$$R_\Delta(\mu \rightarrow 3e) = \frac{1}{2} \left( \frac{c_{e\mu} c_{ee}}{g^2} \right)^2 \left( \frac{m_W}{M_\Delta} \right)^4 \quad (115)$$

which, combined with the experimental limit, gives the following bound

$$\frac{M_\Delta}{\sqrt{c_{e\mu} c_{ee}}} \geq 3.2 \times 10^4 GeV \quad (116)$$

Before closing this section, we should recall that, lepton flavor violating processes play a crucial role in theories proposed to solve the hierarchy problem. One such example is given by the theories of dynamical symmetry breaking (Technicolor (TC) [72] and extended Technicolor (ETC) [73].) Flavor changing reactions, both in leptonic as well as in the quark sector, are found to be incompatible with the experimental limits in these theories. It has been shown recently [74] that, fixed point or walking technicolor theories can solve



the problem of large flavor violation and bring it down to the experimentally allowed region.

The most interesting theory, which can solve the hierarchy problem of course, is the theory of Supersymmetry. Lepton flavor violating reactions are always present in this case, receiving new contributions from the supersymmetric partners and renormalization effects. These, will be discussed in the next section.

### **3.3. Supersymmetric extensions of the standard model**

One of the most important extensions of the standard theory is the minimal supersymmetric standard model (MSSM). Although there is no experimental evidence of supersymmetry as yet, it is a common belief that supersymmetry and supergravity play an important role in the theory of elementary particles. The main motivation for the incorporation of supersymmetry in the fundamental theory of interactions is the natural solution of the hierarchy problem [46]. Moreover, supersymmetry seems to play important role in other issues of the unification program. SUSY models predict a longer life-time for proton, they play crucial role in models with inflation, while they predict exact unification [75] of the three gauge couplings consistent with the LEP-data. Furthermore, superstring theories [76], which appear today as the only candidates for a consistent theory unifying all fundamental interactions, result to an effective supersymmetric theory in low energy.

Flavor changing neutral currents [77, 78, 79, 80, 81] present one of the most important tests of all these low energy supersymmetric theories. In the following subsections we are going to present a brief review of the effects of the new sources of flavor violation in the context of the above theories.

**3.3.1. Minimal Supersymmetric Standard Model.** In the minimal supersymmetric extension of the standard model one can write down the following Yukawa couplings,

$$\mathcal{W} = \lambda_u Q \bar{H} U^C + \lambda_d Q \bar{H} D^C + \lambda_e L \bar{H} E^C \quad (117)$$

where  $Q, D^C, E^C$ , are the usual superfields which accommodate quarks and leptons. The potential is given by

$$\mathcal{V} = \sum_i \left| \frac{\partial \mathcal{W}}{\partial \varphi_i} \right|^2 + m_{3/2}^2 \sum_i |\varphi_i|^2 + A(\mathcal{W} + \mathcal{W}^*) + B \left( \left| \frac{\partial \mathcal{W}}{\partial \varphi_i} \right| \varphi_i + c.c. \right) \quad (118)$$

where  $m_{3/2}$  the gravitino mass and  $A, B$  are the scalar mass parameters depending on the details of the supersymmetry breaking. Then, the s-lepton  $6 \otimes 6$  matrix in the basis  $(\tilde{e}, \tilde{e}^*)$  takes the form

$$\begin{pmatrix} m_{3/2}^2 I + m_e^\dagger m_e & \bar{A}^* m_e^\dagger \\ \bar{A} m_e & m_{3/2}^2 I + m_e^\dagger m_e \end{pmatrix} \quad (119)$$

where  $m_e$  is the usual lepton mass matrix and  $\bar{A} = A + 2B$ . In this approximation, the lepton and s-lepton mixing mass matrices are similar which, in turn, implies that  $S_e^\dagger S_{\tilde{e}} = 1$ . This means that, there is no lepton flavor violation induced.

One can also show that, in this model there is no contribution from neutral intermediate particles, since at this level either they remain massless (neutrinos) or degenerate (s-neutrinos).

The inclusion of the isosinglet right-handed neutrino,  $N^C = (N_L^C; \tilde{N}_C)$ , can lead to additional terms in the superpotential of the form

$$W_1 = \lambda_N L \bar{H} N^C + \frac{1}{2} M_N N^C N^C \quad (120)$$

Now, the  $6 \otimes 6$  neutrino mass matrix can have both Dirac and isosinglet majorana mass terms. The corresponding  $12 \otimes 12$  neutral s-lepton mass matrix becomes analogous to that of the neutrinos. Even though  $S_e^\dagger S_\nu$  is non-diagonal, lepton flavor violating processes are suppressed due to the fact that s-neutrinos are degenerate [78].

The above results are modified if one goes beyond the tree level and includes radiative corrections arising from the first term of the superpotential (eq. (120)) taking into account renormalization effects [80, 81, 82] and considering

Figure 3: Radiative contributions to the  $L_i$  masses.

radiative contributions to the scalar masses (Squarks, Sleptons etc,) at the one loop level. Indeed, let us assume that  $L_i$  are the quark and lepton superfields and that the MSSM is extended with the inclusion of  $X, Y, Z$  additional superfields [80]. Then, one gets the additional Yukawa couplings

$$W' = \lambda_{ij} L_i L_j X + \lambda'_i L_i Y Z \quad (121)$$

The additional terms create the diagrams of fig. 3, which lead to radiative contributions of the type

$$\Delta M_{L_i}^2 \propto \lambda_{ij} \lambda_{ij} + \lambda'_i \lambda'_i \quad (122)$$

where  $\lambda_{ij}$  is a  $3 \otimes 3$  matrix in generation space, while  $\lambda'_i$  is a column vector. Now, if the fields  $X, Y, Z$  are light compared to the Plank scale, then the above corrections are proportional to a large logarithmic factor  $\ln(M_{Pl}/M_{X,Y,Z})$  and the contributions to the scalar masses are significant. The origin of flavor violation here lies in the fact that, due to these contributions, it is no longer possible to diagonalize fermion and s-fermion mass matrices simultaneously.

Let us consider the above effects in the case of s-leptons. In the presence of the right handed neutrino, the mass matrices discussed in the previous section, receive contributions which modify the tree level results as follows

$$m_{\bar{e}} m_e^\dagger = m_{3/2}^2 I + m_e m_e^\dagger + c m_{\nu_D} m_{\nu_D}^\dagger \quad (123)$$

where  $m_{\nu_D}$  is the Dirac type neutrino mass matrix as before. The coefficient  $c = c(t)$  is scale dependent ( $t = \ln \mu$ ) and includes the running from the Plank scale down to the scale where the right-handed neutrino acquires a mass  $M_N \sim \mathcal{O}(M_{GUT})$ , thus  $c$  is proportional to  $\ln(M_{Pl}/M_{GUT})$ . Because the dependence of the corrections on the scale is logarithmic, and the range  $M_{Pl} - M_{GUT}$  is not very large, the corrections inducing the flavor mixing in the charged s-lepton mass matrix can be written as

$$(\Delta m_{\tilde{e}}^2)_{ij} \approx \left[ \frac{3m_{3/2}^2 + A}{(2\pi v \sin \beta)^2} \ln \frac{M_{Pl}}{M_{GUT}} \right] \times (V^* m_{\nu_D}^\delta V^T)_{ij} \quad (124)$$

where  $v = 246 GeV$  and  $\beta = \tan^{-1}(< \bar{H} > / < H >)$ .  $m_{\nu_D}^\delta$  is the diagonalized Dirac neutrino mass matrix at the GUT scale.

The most natural candidate models, that the above analysis can apply, are the models derived from the superstring [83, 84, 85, 86]. Some particular cases will be considered in the next section.

**3.3.2. Flavor violation in Superstring Models.** One of the most promising avenues beyond local supersymmetry is the theory of superstrings [76]. Indeed, superstring theory is the best candidate for a consistent unification of all fundamental forces including gravity. On the other hand, Grand Unified Theories are incorporated naturally in the superstring scenario. Realistic string models suggest that, all gauge interactions should unify within a simple gauge group, with a common gauge coupling  $g_{String}$  at the String unification scale  $M_{String}$ , which is found relatively high and close to the Plank scale [87]

$$M_{String} \approx g_{String} \times M_{Pl} \sim 5. \times 10^{17} GeV \quad (125)$$

On the other hand, renormalization group calculations have indicated that minimal supersymmetric Grand Unified Theories (SUSY-GUTs) are in agreement with the precision LEP data when the SUSY-GUT scale  $M_G$  is taken close to  $M_G \approx 10^{16} GeV$  [75]. Thus, the minimal supersymmetric standard model (MSSM) cannot probably be derived directly at the string scale; the above discrepancy between the two scales, would rather suggest that, either the MSSM should be obtained through the spontaneous breaking of some intermediate GUT-like gauge group which breaks at the scale  $M_G$ , or some extra matter fields are needed to modify the gauge coupling running.

Thus, string motivated non-minimal extensions of the MSSM, contain additional Yukawa interactions, which may in principle lead to interesting enhancement of flavor changing neutral processes. In particular, the radiatively induced lepton flavor violations, discussed in the previous section, occur naturally in this kind of models.

In order to be specific and give some quantitative results, we will concentrate on some realistic string constructions discussed extensively in the literature. These attempts have been made in the context of the free fermionic formulation of the four dimensional superstrings, and led to the construction of three types of models. Two of them are characterized by an intermediate GUT scale based on the symmetries  $SU(5) \otimes U(1)$  [84] and  $SU(4) \otimes O(4)$  [85], while there is the third type of models [86], where the original string symmetry breaks down to the standard model symmetry times some additional  $U(1)$  factors.

In the case of flipped  $SU(5)$  model [84], the superpotential of the minimal model (assuming invariance under  $H \rightarrow -H$ ) reads

$$\begin{aligned} \mathcal{W} = & \lambda_1^{ij} F_i F_j h + \lambda_2^{ij} F_i \bar{f}_j \bar{h} + \lambda_3^{ij} \bar{f}_i l_j^c h + \lambda_4 H H h \\ & + \lambda_5 \bar{H} \bar{H} \bar{h} + \lambda_6^{ia} F_i \bar{H} \phi_a + \lambda_7 h \bar{h} \phi_0 + \mu_{ab} \phi_a \phi_b \end{aligned} \quad (126)$$

The  $F_i + \bar{f}_i + l_i^c$  ( $i = 1, 2, 3$ ) are the three generations of **10**,  $\bar{\mathbf{5}}$  and singlet representations of  $SU(5)$  that accommodate the light matter particles of the Standard Model,  $H$  and  $\bar{H}$  are **10** and  $\bar{\mathbf{10}}$  Higgs representations,  $h$  and  $\bar{h}$  are **5** and  $\bar{\mathbf{5}}$  Higgs representations, and the  $\phi_0, \phi_a$  ( $a = 1, 2, 3$ ) are auxiliary singlet fields. The first 3 terms in the superpotential eq. (126) give masses to the charge 2/3 quarks  $u_i$ , charge -1/3 quarks  $d_i$  and charged leptons  $l_i$ , respectively. The next two terms split the light Higgs doublets from their heavy colour triplet partners in a natural way. The sixth term provides a large element in the see-saw neutrino mass matrix, and the term  $\lambda_7 h \bar{h} \phi_0$  gives the traditional Higgs mixing parameter. Under the same assumptions discussed in the previous sections, the induced mixing in the s-lepton mass matrix, due to renormalization group running in the range  $M_{Pl} - M_{GUT}$ , is [88, 89]

$$(\delta m_e^2)_{ij} \approx \left[ \frac{3m_{3/2}^2 + A}{(\pi v \sin \beta)^2} \ln \frac{M_{SU}}{M_{GUT}} \right] \times (V^* m_u^\delta V^T)_{ij} \quad (127)$$

where, due to the fact that up-quark and Dirac neutrino masses arise from the same superpotential term, we have substituted  $m_{\nu_D}^\delta$  with  $m_u^\delta$ , which is the diagonal up-quark mass matrix at the GUT scale.

The diagonalizing unitary matrix  $V \equiv S_e^\dagger S_\nu$  is unknown, due to the fact that  $m_e$  and  $m_d$  mass matrices are unrelated in the flipped  $SU(5)$ . Thus, it is possible that, the mixing may enhance the flavor changing reactions in this model.

In the case of the  $SU(4) \otimes O(4) \equiv SU(4) \otimes SU(2)_L \otimes SU(2)_R$  string model, the lepton and KM-mixing matrices are related due to the fact that leptons and down quarks receive masses from the same superpotential term. The superpotential of the model under a  $Z_2$  symmetry  $H \leftrightarrow -H$  can be written as follows

$$\begin{aligned} \mathcal{W} = & \lambda_1 F_L \bar{F}_R h + \lambda_2 \bar{F}_R H \phi_i + \lambda_3 H H D + \lambda_4 \bar{H} \bar{H} D \\ & + \lambda_5 \bar{F}_R \bar{F}_R D + \lambda_6 F_L F_L D + \lambda_7 \phi_0 h h + \lambda_8 \phi^3 \end{aligned} \quad (128)$$

The phenomenological implications of the above superpotential terms have been discussed extensively in previous works [90].

The minimal supersymmetric version of the model includes three generations of quarks and leptons, which are accommodated in  $F_L + \bar{F}_R \equiv (4, 2, 1) + (\bar{4}, 1, 2)$  representations of the  $SU(4) \otimes SU(2)_L \otimes SU(2)_R$  symmetry. One needs at least one pair of  $H + \bar{H} \equiv (4, 1, 2) + (\bar{4}, 1, 2)$  Higgses to realize the first symmetry breaking  $SU(4) \times SU(2)_R \rightarrow SU(3) \times U(1)_{B-L}$ , and one  $h \equiv (1, 2, 2)$  higgs field to provide the two Weinberg-Salam doublets after the first symmetry breaking. In addition, a sextet field  $D \equiv (6, 1, 1)$  is needed to produce a pair of colored triplets  $3 + \bar{3}$ , which are going to combine with the uneaten  $d_H^c$  and  $\bar{d}_H^c$ , to form two superheavy massive states and avoid fast proton decay.

Thus, in the case of the minimal version of this model, one obtains the mass relations  $m_u = m_{\nu_D}$  and  $m_e = m_d$  at the GUT scale, which in turn imply that the lepton and KM mixing matrices are also equal at  $M_{GUT}$ . This fact leads to a definite prediction for the above processes. Due to the small mixing angles, however, there is a significant suppression of flavor violating processes.

One novel feature of both models is the generalized see-saw mechanism [84, 85, 91]. Indeed, three of the singlet fields introduced, are used to realize the

see-saw mechanism together with the left and right handed neutrinos. The see-saw matrix in the basis  $(\nu_i, N_i^C, \phi_m)$  takes the form

$$m_\nu = \begin{pmatrix} 0 & m_{\nu_D} & 0 \\ m_{\nu_D} & M^{rad} & M_{\nu^c, \phi} \\ 0 & M_{\nu^c, \phi} & \mu_\phi \end{pmatrix} \quad (129)$$

where it is understood that, all entries in eq. (129) represent  $3 \times 3$  matrices.

The  $m_{\nu_D}$ ,  $M_{\nu^c, \phi}$ ,  $\mu_\phi$  submatrices arise from the trilinear superpotential terms of the above models. The contribution  $M^{rad}$  is a majorana type mass matrix for the right handed neutrino and usually arises from some higher order non-renormalizable terms. In principle, there are many arbitrary parameters in eq. (129), but within the context of some recently proposed fermion mass matrix Ansätze, as well as under some natural assumptions [92, 93], it is possible to reduce the arbitrariness and obtain definite predictions. In what follows, we are going to use the results of a fermion mass matrix Ansatz proposed in the context of the above string derived models to estimate the renormalization effects on the various flavor violating processes.

**3.3.3. Flavor violating processes.** *i)* We start again with the simplest decay  $\mu \rightarrow e\gamma$ . The diagrams are presented in fig. 4. We can parametrize the amplitude with the functions  $f_{M1}, f_{E1}$  in the same way as in sect. 3.1. Here they are given by

$$f_{M1} = -f_{E1} = -\frac{1}{2} \tilde{\eta} \alpha^2 \frac{m_\mu^2}{\tilde{m}_\alpha^2} f(x) \quad (130)$$

where  $\tilde{\eta}$  is the corresponding flavor violating quantity defined as follows

$$\tilde{\eta} = -\frac{(\delta \tilde{m}_{\tilde{e}\tilde{e}^*}^2)_{12}}{\tilde{m}_a^2} \quad (131)$$

and  $(\delta \tilde{m}_{\tilde{e}\tilde{e}^*}^2)_{12}$  is determined by the analysis of the previous section while  $\tilde{m}_a$  is the mass of the heaviest sparticle circulating in the loop. If  $\tilde{m}_a^2 = m_e^2$  for example, it is given by [94, 95]

Figure 4:  $\mu \rightarrow e\gamma$  in supersymmetric theories.

$$\tilde{m}_{\tilde{e}}^2 = m_{3/2}^2 + C_{\tilde{e}}(t)m_{1/2}^2 \quad (132)$$

where  $C_{\tilde{e}}(t)$  is a function of the scale  $t = \ln\mu$ , and for  $\mu \sim m_W$ ,  $C_{\tilde{e}} \approx 0.50$ . The function  $f(x)$  of eq. (130), depends on the ratio  $x = m_{\tilde{\gamma}}/\tilde{m}_a$ ,  $m_{\tilde{\gamma}}$  being the photino mass, and is given by

$$f(x) = \frac{1}{12(1-x)^4} \{1 + 2x^3 + 3x^2 - 6x - 6x^2 \ln x\}, x = \frac{m_{\tilde{\gamma}}^2}{\tilde{m}_\alpha^2} \quad (133)$$

Now, the branching ratio  $R_{e\gamma}$  takes the form

$$R_{e\gamma} = \frac{6\pi}{\alpha} \frac{|f_{E1}|^2 + |f_{M1}|^2}{(G_F m_\mu^2)^2} = |\tilde{\eta}|^2 R_0 \quad (134)$$

where  $R_0$  is defined through

$$R_0 = \frac{3\pi\alpha^3 |f|^2}{(G_F m_\alpha^2)^2} \quad (135)$$

As an application, we use the results of ref. [93] for the leptonic mixing angles. Taking as an example the initial condition at the GUT scale,  $m_{3/2} = m_{1/2}$ , we can obtain the bound  $m_{3/2} \geq 25 \text{ GeV}$ .

ii) The decay  $\mu \rightarrow 3e$  is treated similarly. The corresponding diagrams are shown in fig. 5. We find

$$R_{3e} = \frac{|\tilde{\eta}|^2 \alpha^4}{(G_F m_\alpha^2)^2} \frac{1}{2} \left\{ \left( 16 \ln \frac{m_\mu}{m_e} - \frac{26}{3} \right) f^2 - 12fg + 3g^2 + 2f_b^2 + 4gf_b - 8ff_b \right\} \quad (136)$$

with



Figure 5: Diagrams for the  $\mu \rightarrow 3e$  process in supersymmetric theories.

$$g(x) = \frac{1}{36(1-x)^4} \{2 - 11x^3 + 18x^2 - 9x + 6x^2 \ln x\} \quad (137)$$

$$f_b(x) = \frac{1}{8(1-x)^4} \{1 - 5x^2 + 4x + 2x(x+2) \ln x\}, x = \frac{m_{\tilde{\gamma}}^2}{\tilde{m}_\alpha^2} \quad (138)$$

Comparing  $\mu \rightarrow e\gamma$  with  $\mu \rightarrow 3e$  we get

$$\frac{R_{3e}}{R_{e\gamma}} \simeq \frac{a}{24\pi} \left\{ 16 \ln \frac{m_\mu}{m_e} - \frac{26}{3} - \frac{61}{6} \right\} \sim 6.4 \times 10^{-3} \quad (139)$$

iii) Let us now discuss the  $(\mu - e)$  conversion. The amplitude for this process is given by

$$\mathcal{M} = \left\{ \frac{j_{(1)}^\lambda J_\lambda^{(1)}}{q^2} + \frac{j_{(2)}^\lambda J_\lambda^{(2)}}{m_\mu^2} \zeta \right\} \quad (140)$$

where the first term corresponds to the photonic and the second to the non-photonic contributions arising from the box diagrams (see fig. 6) and  $\zeta = m_{3/2}^2/m_u^2$ . We find that

$$j_{(1)}^\lambda = \bar{u}(p_1) (f_{M1} + \gamma_5 f_{E1}) i\sigma^{\lambda\nu} \frac{q_\nu}{m_\mu} + \frac{q^2}{m_\mu^2} (f_{E0} + \gamma_5 f_{M0}) \gamma^\nu \left( g_{\lambda\nu} - \frac{q^\lambda q^\nu}{q^2} \right) \quad (141)$$

$$J_\lambda^{(1)} = \bar{N} \gamma_\lambda \frac{1 + \tau_3}{2} N, \quad N = Nucleon \quad (142)$$

For the box diagrams we obtain

$$j_{(2)}^\lambda = \bar{u}(p_1) \gamma^\lambda \frac{1}{2} (\tilde{f}_V + \tilde{f}_A \gamma_5) u(p_\mu) \quad (143)$$

Figure 6: Diagrams for the  $(\mu^- \rightarrow e^-)$  conversion in supersymmetric theories.

$$J_\lambda^{(2)} = \bar{N} \gamma_\lambda \frac{1}{2} [(3 + \beta f_V \tau_3) - (f_V + f_A \beta \tau_3) \gamma_5] N \quad (144)$$

where  $\beta = \beta_0/\beta_1$ , with

$$\beta_0 = \frac{4}{9} + \frac{1}{9} \frac{m_u^2}{m_d^2} \quad (145)$$

$$\beta_1 = \frac{4}{9} - \frac{1}{9} \frac{m_u^2}{m_d^2} \quad (146)$$

Furthermore,

$$f_{E0} = -f_{M0} = -\frac{1}{2} \tilde{\eta} \alpha^2 g(x) \frac{m_\mu^2}{m_{3/2}^2} \quad (147)$$

$$\tilde{f}_V = -\tilde{f}_A = -\frac{\beta_0}{2} \tilde{\eta} \alpha^2 f_b(x) \frac{m_\mu^2}{m_{3/2}^2} \quad (148)$$

It is quite hard to write down general expressions containing both photonic as well as non-photonic diagrams (see sect. 4.1). For the coherent process, however, one can write down the  $(\mu - e)$  conversion rate as follows

$$R_{eN} = \frac{1}{(G_F m_\mu^2)^2} \left\{ \left| \frac{m_\mu^2}{q^2} f_{M1} + f_{E0} + \frac{1}{2} \kappa \tilde{f}_V \right|^2 + \left| \frac{m_\mu^2}{q^2} f_{E1} + f_{M0} + \frac{1}{2} \kappa \tilde{f}_A \right|^2 \right\} \gamma_{ph} \quad (149)$$

where

$$\kappa = \left( 1 + \frac{N}{Z} \frac{3 - \beta}{3 + \beta} \frac{F_N(q)^2}{F_Z(q)^2} \right) \zeta \quad (150)$$

and

$$\gamma_{ph} = \frac{Z|F_Z(q^2)|^2}{6f(A, Z)} \quad (151)$$

$F_{Z,N}(q^2)$  are the nuclear form factors to be discussed later and  $f(A, Z)$  is the Primakoff's function (see eq. (99)). Under some plausible approximations, we can write the  $(\mu - e)$  branching ratio as

$$R_{eN} = \frac{1}{2} \frac{|\tilde{\eta}|^2 \alpha^4}{(G_F m_{3/2}^2)^2} \left\{ f - g + \frac{1}{2} f_b \kappa \right\}^2 \gamma_{ph} \quad (152)$$

Assuming that the photino is the lightest supersymmetric particle, we can take  $x \ll 1$ , to obtain

$$f = \frac{1}{12}, \quad g = \frac{1}{18}, \quad f_b = \frac{1}{8} \quad (153)$$

Thus, the two terms from the photonic contribution tend to cancel and the box diagram dominates. Comparing the  $(\mu^-, e^-)$  conversion with the  $\mu \rightarrow e\gamma$  decay, we get

$$R_{eN} \approx \frac{\alpha}{6\pi} \left( \frac{1}{3} + \frac{3\kappa}{4} \right)^2 \gamma_{ph} R_{e\gamma} \quad (154)$$

Analytic results of the above branching ratios are presented in sect. 5.3.

#### 4. EXPRESSIONS FOR THE BRANCHING RATIO OF $(\mu^-, e^-)$

In this section we will discuss the construction at nuclear level of the effective operators which are responsible for the semileptonic process  $\mu^-(A, Z) \rightarrow e^-(A, Z)^*$  and derive the expressions for the branching ratios of this process. If we assume that the  $A$  nucleons of the nucleus  $(A, Z)$  interact individually with the muon field (impulse approximation), then the needed nuclear matrix elements can be obtained from an effective Hamiltonian  $\Omega$  which arises from that of the free particles, as it is described in the next section.

##### 4.1. The effective Hamiltonian of $(\mu^-, e^-)$ conversion

As we have seen in sect. 2, the effective amplitude for  $(\mu^-, e^-)$  conversion is given by eq. (140). It is not easy to separate the dependence on the nuclear physics from the leptonic form factors. This can only be done for the coherent mode. For the general case however, we will discuss separately the photonic and the non-photonic contributions. For compactness of our notation, we will write the hadronic currents given in eqs. (142), (144) in the general form

$$J_\lambda^{(2)} = \bar{N} \gamma_\lambda [\tilde{g}_V(3 + \beta f_V \tau_3) - \tilde{g}_A(f_V + f_A \beta \tau_3) \gamma_5] N \quad (155)$$

where

$$\tilde{g}_V = \frac{1}{6}, \quad \tilde{g}_A = 0, \quad f_V = 1, \quad \beta = 3 \quad (\text{photonic case}) \quad (156)$$

$$\tilde{g}_V = \tilde{g}_A = \frac{1}{2}, \quad f_V = 1, \quad f_A = 1.24 \quad (\text{non-photonic case}) \quad (157)$$

The parameters  $\beta$ ,  $f_{E0}$ ,  $f_{E1}$ ,  $f_{M0}$ ,  $f_{M1}$ ,  $\tilde{f}_A$ ,  $\tilde{f}_V$  and  $\zeta$  depend, of course, on the model. Thus, in the case of the neutrino mediated processes in left handed theories one finds

$$\zeta = \frac{G_F m_\mu^2}{\sqrt{2}}, \quad \beta_0 = \left\{ \begin{array}{c} 30 \\ 1 \end{array} \right\}, \quad \beta_1 = \left\{ \begin{array}{cc} 25 & (\text{light neutrinos}) \\ 5/6 & (\text{heavy neutrinos}) \end{array} \right\} \quad (158)$$

i.e.  $\beta = 5/6$ . Furthermore

$$\tilde{f}_V = \beta_0 f_1, \quad \tilde{f}_A = \beta_0 f_2 \quad (159)$$

The quantities  $f_1$ ,  $f_2$ , depend on the specific gauge model and are given in the literature [12]. For the supersymmetric models, the corresponding expressions have already been given in sect. 3.2.2.

At the nuclear level the relativistic expression of the hadronic current is not needed. Using the standard non-relativistic limits the relevant nuclear matrix elements involve the operators

$$\Omega_0 = \tilde{g}_V \sum_{j=1}^A (3 + f_V \beta \tau_{3j}) e^{-i \mathbf{p}_e \cdot \mathbf{r}_j}, \quad \Omega = -\tilde{g}_A f_A \sum_{j=1}^A (\xi + \beta \tau_{3j}) \frac{\sigma_j}{\sqrt{3}} e^{-i \mathbf{p}_e \cdot \mathbf{r}_j} \quad (160)$$

where the summation is over all nucleons and  $\mathbf{p}_e$  is the momentum of the outgoing lepton. The factor  $1/\sqrt{3}$  is introduced to make the two matrix elements in the total rate expression equal and compensated by suitable factors elsewhere in the expression. In eq. (160)

$$\xi = f_V/f_A = 1/1.24 \quad (161)$$

Assuming that the kinetic energy of the final nucleus is negligible and taking  $m_e \approx 0$ , we can write the magnitude of the momentum  $\mathbf{p}_e$  of the outgoing electron approximately as

$$p_e = |\mathbf{p}_e| \approx m_\mu - \epsilon_b - (E_f - E_{gs}) \quad (162)$$

where  $E_f, E_{gs}$  are the energies of the final and ground state of the nucleus, respectively.  $m_\mu$  is the muon mass and  $\epsilon_b$  is the binding energy. For coherent processes  $E_f = E_i$  and since  $\epsilon_b$  is relatively small (the biggest value occurs in lead region where  $\epsilon_b \approx 10 \text{ MeV}$ ) [96], we have  $p_e \approx m_\mu \approx 0.53 \text{ fm}^{-1}$ . For incoherent processes  $E_f \neq E_i$  and for a sum-rule approach one can assume a mean energy [63, 64] for the emitted electron corresponding to a mean excitation energy of the nucleus  $\bar{E}$ . In this case  $p_e \approx m_\mu - \bar{E} - \epsilon_b$ .

By expanding the exponential of the operators in eq.(160) in terms of spherical Bessel functions  $j_l(x)$ , we obtain the multipole expansion of the  $(\mu^-, e^-)$  operator, i.e. the following two types of operators  $\hat{T}^{(l,\sigma)J}$

$$\hat{T}_M^{(l,0)J} = \tilde{g}_V \delta_{lJ} \sqrt{4\pi} \sum_{i=1}^A (3 + \beta \tau_{3i}) j_l(qr_i) Y_M^l(\hat{\mathbf{r}}_i) \quad (163)$$

for the vector part, and

$$\hat{T}_M^{(l,1)J} = \tilde{g}_A \sqrt{\frac{4\pi}{3}} \sum_{i=1}^A (\xi + \beta \tau_{3i}) j_l(qr_i) [Y^l(\hat{\mathbf{r}}_i) \otimes \sigma_i]^J_M \quad (164)$$

for the axial vector part. In eqs. (163), (164)  $q$  represents the magnitude of the momentum transferred  $\mathbf{q}$  to the nucleus during the  $(\mu^-, e^-)$  process. In a good approximation  $q \approx p_e$  and thus,  $q$  is given by the energy conservation eq. (162).

#### 4.2. Expressions for the branching ratio of $(\mu^-, e^-)$ conversion

The probability density for converting the bound  $\mu^-$  of a muonic atom to an  $e^-$  with momentum  $\mathbf{p}_e$  is given by the Fermi's golden rule

$$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} \int d\hat{\mathbf{p}}_e \left( \frac{p_e}{m_\mu} \right)^2 |\langle f | \Omega | i, \mu \rangle|^2 \Xi \quad (165)$$

where  $|i, \mu\rangle$  is the initial state of the system: nucleus  $(A, Z) + \mu^-$  and  $|f\rangle$  the final state of the system: nucleus  $(A, Z)^* + e^-$ . The factor  $(p_e/m_\mu)^2$  involves the density of the final states appropriate for normalization of the wave packet representing the outgoing  $e^-$  and  $\hat{\mathbf{p}}_e$  is the unit vector in the direction of the electron momentum. The quantity  $\Xi$  depends on the gauge model (in the photonic case, for example, it coincides with  $\xi_0$  of eq. (97)). The effect of nuclear recoil has been neglected. In the above expression

$$|\langle f | \Omega | i, \mu \rangle|^2 = |\langle f | \Omega_0 | i, \mu \rangle|^2 + 3 |\langle f | \vec{\Omega} | i, \mu \rangle|^2 \quad (166)$$

Of special interest, as we shall extensively discuss below, is the partial rate for the  $gs \rightarrow gs$  transitions i.e. the coherent rate. It has been previously assumed that, the  $1s$  muon wave function varies very little inside the light and medium nuclei [96], i.e. the following approximation has been used

$$|\langle f | \Omega | i, \mu \rangle|^2 = \langle \Phi_{1s} \rangle^2 |\langle f | \Omega | i \rangle|^2 \quad (167)$$

where

$$\langle \Phi_{1s} \rangle^2 \equiv \frac{\int d^3x |\Phi_\mu(\mathbf{x})|^2 \rho(\mathbf{x})}{\int d^3x \rho(\mathbf{x})} \quad (168)$$

In the latter definition,  $\Phi_\mu(\mathbf{x})$  is the muon wave function and  $\rho(\mathbf{x})$  is the nuclear charge density. To a good approximation it has been found [96]

$$\langle \Phi_{1s} \rangle^2 = \frac{\alpha^3 m_\mu^3}{\pi} \frac{Z_{eff}^4}{Z} \quad (169)$$

( $\alpha$  is the fine structure constant) i.e. the deviation from the behaviour of the wave function at the origin has been taken into account by  $Z_{eff}$ . The above approximation was first used for the  $(\mu^-, \nu_\mu)$  [63, 64] and afterwards in analogy to muon capture in the  $(\mu^-, e^-)$  process [97, 98, 31]. Recently [103], it had been found that eq. (167) is not very accurate in ordinary muon capture. We will therefore present results, both, with and without this approximation.

The branching ratio of the total  $(\mu^-, e^-)$  conversion rate divided by the total muon capture rate, assuming the approximation of eq. (167) for both  $(\mu, e)$  and  $(\mu, \nu)$  processes and only photonic or non-photonic mechanisms, takes the simple form

$$R_{eN} = \frac{\Gamma(\mu, e)}{\Gamma(\mu, \nu)} = \rho \gamma \quad (170)$$

where the quantity  $\rho$  contains all the nuclear dependence of  $R_{eN}$  and  $\gamma$  contains the elementary particle physics. We mention that  $R_{eN}$  is the quantity provided by experiments. Obviously, the effect of the approximation eq. (167) on the branching ratio is expected to be negligible.

Another interesting quantity in the study of the  $\mu - e$  process is the ratio of the coherent  $(\mu^-, e^-)$  rate,  $\Gamma_{coh} = \Gamma_{gs \rightarrow gs}$ , to the total  $(\mu^-, e^-)$  rate,  $\Gamma_{tot} = \sum_f \Gamma_{i \rightarrow f}$  for all final states  $|f\rangle$ , i.e.

$$\eta = \frac{\Gamma_{coh}(\mu^- \rightarrow e^-)}{\Gamma_{total}(\mu^- \rightarrow e^-)} \quad (171)$$

This will be discussed in detail below.

#### 4.2.1. Coherent $(\mu^-, e^-)$ conversion .

In the case of the coherent process, i.e. ground state to ground state ( $0^+ \rightarrow 0^+$ ) transitions only the vector component of eq. (166) contributes and one obtains

$$\langle f | \Omega_0 | i, \mu \rangle = \tilde{g}_V (3 + f_V \beta) \tilde{F}(q^2) \quad (172)$$

where  $\tilde{F}(q^2)$  is the matrix element involving the ground state

$$\tilde{F}(q^2) = \int d^3x \{ \rho_p(\mathbf{x}) + \frac{3 - f_V \beta}{3 + f_V \beta} \rho_n(\mathbf{x}) \} e^{-i\mathbf{q} \cdot \mathbf{x}} \Phi_\mu(\mathbf{x}) \quad (173)$$

( $\rho_p(\mathbf{x})$ ,  $\rho_n(\mathbf{x})$  represent the proton, neutron densities normalized to Z and N, respectively).  $\tilde{F}(q^2)$  can also be written as

$$\tilde{F}(q^2) = \tilde{F}_p(q^2) + \frac{3 - f_V \beta}{3 + f_V \beta} \tilde{F}_n(q^2) \quad (174)$$

where

$$\tilde{F}_{p,n}(q^2) = \int d^3x \rho_{p,n}(\mathbf{x}) e^{-i\mathbf{q} \cdot \mathbf{x}} \Phi_\mu(\mathbf{x}) \quad (175)$$

In eqs. (174), (175) we have not made use of the eq. (167). Thus, the coherent rate can be obtained from eq. (172) by first calculating  $\tilde{F}(q^2)$  from eq. (173) for a given muon wave function and a given nuclear density distribution. In ref. [99] the muon wave function was obtained by solving numerically the Schrödinger equation taking into consideration the effects of vacuum polarization and finite nuclear size. Weinberg and Feinberg [97] used for the coherent rate the expression of eq. (167) and estimated the quantity  $Z_{eff}$  as in ref. [96]. Shanker [98] using Fermi nuclear distribution included an additional correction interference term. Using eqs. (167) and (169), for the coherent rate one obtains

$$|\tilde{F}(q^2)|^2 \approx \frac{\alpha^3 m_\mu^3}{\pi} \frac{Z_{eff}^4}{Z} |ZF_Z(q^2) + \frac{3 - f_V \beta}{3 + f_V \beta} NF_N(q^2)|^2 \quad (176)$$

with  $F_Z$  ( $F_N$ ) the proton (neutron) nuclear form factors

$$F_Z(q^2) = \frac{1}{Z} \int d^3x \rho_p(\mathbf{x}) e^{-i\mathbf{q} \cdot \mathbf{x}}, \quad F_N(q^2) = \frac{1}{N} \int d^3x \rho_n(\mathbf{x}) e^{-i\mathbf{q} \cdot \mathbf{x}} \quad (177)$$

These nuclear form factors can be calculated by using various models i.e. shell model [100], quasi-particle RPA [101] etc., or can be obtained from



experimental data whenever possible [102]. The branching ratio in this approximation takes the form

$$R_{eN} = \frac{\Xi}{(G_F^2 m_\mu^2)^2} \tilde{g}_V^2 \left[ 1 + \frac{3 - f_V \beta}{3 + f_V \beta} \frac{F_N(q^2)}{F_Z(q^2)} \right]^2 \gamma_{ph}. \quad (178)$$

with  $\gamma_{ph}$  defined in eq. (151). The validity of the approximation of eq. (167) will be discussed in sect. 5.3. The parameters  $F_N$ ,  $F_Z$ ,  $\tilde{F}_p$ ,  $\tilde{F}_n$  and  $Z_{eff}$  for various nuclear systems appear in *table 1*. The corresponding widths in arbitrary units are given in *table 2*.

*Table 1. Parameters needed for calculations of the coherent  $(\mu^-, e^-)$  conversion matrix elements with: i) the exact muon wavefunction,  $\tilde{F}_p$  and  $\tilde{F}_n$  of eq. (175), and ii) two-parameter Fermi distribution, proton and neutron form factors,  $F_Z$  and  $F_N$ . The electron momentum  $p_e$  and the effective charge  $Z_{eff}$  of a set of nuclei covering the hole periodic table are also shown.*

A	Z	$\tilde{F}_p[fm^{-3/2}]$	$\tilde{F}_n[fm^{-3/2}]$	$F_Z$	$F_N$	$p_e(MeV/c)$	$Z_{eff}$
12	6	$0.86 \cdot 10^{-2}$	$0.86 \cdot 10^{-2}$	0.77	0.77	105.067	5.74
24	12	$0.37 \cdot 10^{-1}$	$0.36 \cdot 10^{-1}$	0.65	0.64	105.017	10.81
27	13	$0.45 \cdot 10^{-1}$	$0.45 \cdot 10^{-1}$	0.66	0.62	104.976	11.62
32	16	$0.66 \cdot 10^{-1}$	$0.63 \cdot 10^{-1}$	0.62	0.59	104.781	13.81
40	20	$0.99 \cdot 10^{-1}$	$0.92 \cdot 10^{-1}$	0.58	0.54	104.449	16.47
44	20	$0.97 \cdot 10^{-1}$	$0.11 \cdot 10^0$	0.57	0.55	104.464	16.43
48	22	$0.11 \cdot 10^0$	$0.13 \cdot 10^0$	0.55	0.52	104.258	17.61
63	29	$0.17 \cdot 10^0$	$0.19 \cdot 10^0$	0.49	0.46	103.474	21.22
90	40	$0.26 \cdot 10^0$	$0.29 \cdot 10^0$	0.42	0.38	101.951	25.69
112	48	$0.29 \cdot 10^0$	$0.36 \cdot 10^0$	0.36	0.33	100.749	27.86
208	82	$0.42 \cdot 10^0$	$0.57 \cdot 10^0$	0.25	0.22	95.125	33.81
238	92	$0.40 \cdot 10^0$	$0.57 \cdot 10^0$	0.20	0.18	93.591	34.36

*Table 2. Coherent widths (in arbitrary units) for the photonic and non pho-*

tonic mechanisms: (i) with the exact muon wave function,  $\Phi_\mu(\mathbf{x})$  (ii) with  $\langle \Phi_\mu(\mathbf{x}) \rangle$  in the approximation of  $Z_{eff}$ . The ratios of the width in case (i) to the width of ordinary muon capture with arbitrary normalization are also given.

non-photonic mechanism ( $\beta = 5/6$ )				photonic mechanism ( $\beta = 3$ )			
A	Z	With $\Phi_\mu(\mathbf{x})$	With $\langle \Phi_\mu(\mathbf{x}) \rangle$	With Ratio	With $\Phi_\mu(\mathbf{x})$	$\langle \Phi_\mu(\mathbf{x}) \rangle$	Ratio
12	6	$0.52 \cdot 10^{-4}$	$0.51 \cdot 10^{-4}$	0.14	$0.21 \cdot 10^{-4}$	$0.21 \cdot 10^{-4}$	0.058
24	12	$0.91 \cdot 10^{-3}$	$0.90 \cdot 10^{-3}$	0.19	$0.38 \cdot 10^{-3}$	$0.37 \cdot 10^{-3}$	0.078
27	13	$0.14 \cdot 10^{-2}$	$0.14 \cdot 10^{-2}$	0.20	$0.56 \cdot 10^{-3}$	$0.55 \cdot 10^{-3}$	0.083
32	16	$0.29 \cdot 10^{-2}$	$0.28 \cdot 10^{-2}$	0.22	$0.12 \cdot 10^{-2}$	$0.12 \cdot 10^{-2}$	0.093
40	20	$0.64 \cdot 10^{-2}$	$0.62 \cdot 10^{-2}$	0.26	$0.27 \cdot 10^{-2}$	$0.26 \cdot 10^{-2}$	0.110
44	20	$0.72 \cdot 10^{-2}$	$0.69 \cdot 10^{-2}$	0.40	$0.26 \cdot 10^{-2}$	$0.25 \cdot 10^{-2}$	0.150
48	22	$0.94 \cdot 10^{-2}$	$0.91 \cdot 10^{-2}$	0.36	$0.35 \cdot 10^{-2}$	$0.34 \cdot 10^{-2}$	0.140
63	29	$0.21 \cdot 10^{-1}$	$0.19 \cdot 10^{-1}$	0.36	$0.78 \cdot 10^{-2}$	$0.72 \cdot 10^{-2}$	0.140
90	40	$0.47 \cdot 10^{-1}$	$0.42 \cdot 10^{-1}$	0.54	$0.17 \cdot 10^{-1}$	$0.16 \cdot 10^{-1}$	0.200
112	48	$0.62 \cdot 10^{-1}$	$0.54 \cdot 10^{-1}$	0.62	$0.22 \cdot 10^{-1}$	$0.19 \cdot 10^{-1}$	0.220
208	82	$0.13 \cdot 10^0$	$0.89 \cdot 10^{-1}$	0.98	$0.41 \cdot 10^{-1}$	$0.29 \cdot 10^{-1}$	0.310
238	92	$0.12 \cdot 10^0$	$0.73 \cdot 10^{-1}$	0.92	$0.35 \cdot 10^{-1}$	$0.22 \cdot 10^{-1}$	0.280

**4.2.2. Incoherent ( $\mu^-, e^-$ ) conversion.** As we have mentioned in sect. 2.3, from the experimental point of view, the coherent contribution to the ( $\mu^-, e^-$ ) conversion is the most interesting. It is, however, important to know what fraction of the total rate goes into the coherent mode and how this varies with A and Z. In this section we will evaluate the incoherent contribution and will compare it to that of the coherent channel.

For the calculation of the contributions leading to the exited states many methods exist which involve various approximations. Those which are based

on the approximation inserted by eq. (167) will be discussed in detail in sect. 5. In this section we will elaborate on a method recently developed by Chiang *et al.*, [99] and which uses the exact muon wave function in eq. (165). This method, known as "nuclear matter mapped into nuclei by means of a local density approximation", is accurate and leads to a good reproduction of the  $\mu^-$  capture rates over the periodic table once the proper renormalization of the weak currents is considered.

In the ordinary muon capture only the incoherent channel is open. In this process the total rate  $\Gamma_{\mu c}$  is given by [103]

$$\Gamma_{\mu c} = \int d^3r |\Phi_\mu(\mathbf{x})|^2 \Gamma(\rho_n(\mathbf{x}), \rho_p(\mathbf{x})) \quad (179)$$

where  $\Gamma(\rho_n, \rho_p)$  is the width in an infinite slab of nuclear matter with neutron and proton densities  $\rho_n, \rho_p$ , respectively. This width can be obtained by means of the Lindhard function as

$$\Gamma(\rho_n, \rho_p) = -2 \int \frac{d^3p_\nu}{(2\pi)^3} \Pi_i \frac{2m_i}{2E_i} \bar{\Sigma} \Sigma |T|^2 \text{Im} \bar{U}_{p,n}(p_\mu - p_\nu) \quad (180)$$

where  $T$  is the transition matrix, i.e. the amplitude associated with the elementary process  $\mu^- p \rightarrow n \nu_\mu$  and  $m_i, E_i$  are the masses, energies of the particles involved in this reaction. The Lindhard function  $\bar{U}$  corresponds to ph excitations of the p-n type and is given by (see e.g. ref. [104, 106] for its usage)

$$\bar{U}_{1,2}(q) = 2 \int \frac{d^3p}{(2\pi)^3} \frac{n_1(\mathbf{p})[1 - n_2(\mathbf{q} + \mathbf{p})]}{q^0 + E_1(\mathbf{p}) - E_2(\mathbf{q} + \mathbf{p}) + i\epsilon} \quad (181)$$

with  $n_1(\mathbf{q}')$  and  $n_2(\mathbf{q})$  the integral (0 or 1) occupation probabilities of the neutron and proton, respectively.

In an analogous formalism, the incoherent ( $\mu^-, e^-$ ) conversion rate,  $\Gamma_{inc}(\mu^- \rightarrow e^-) = \sum_{f \neq gs} \Gamma_{i \rightarrow f}$ , can be expressed as follows

$$\begin{aligned} \Gamma_{inc}(\mu^- A \rightarrow e^- X) &= -2 \int d^3x |\Phi_\mu(\mathbf{x})|^2 \int \frac{d^3p_e}{(2\pi)^3} \Pi_i \frac{2m_i}{2E_i} \\ &\times [\bar{\Sigma} \Sigma |T|^2 (\mu^- p \rightarrow e^- p) \text{Im} \bar{U}_{p,p}(p_\mu - p_e) \\ &+ \bar{\Sigma} \Sigma |T|^2 (\mu^- n \rightarrow e^- n) \text{Im} \bar{U}_{n,n}(p_\mu - p_e)] \quad (182) \end{aligned}$$

The two terms in the brackets of eq. (182) result from the character of the  $(\mu^-, e^-)$  operator to be of charge conserving type.

One can separate the nuclear dependence from the dependence on the elementary sector in the two processes,  $(\mu^-, \nu_\mu)$  and  $(\mu^-, e^-)$ , by factorizing outside the integrals in eqs. (179), (182) an average value of the quantity  $\bar{\Sigma}\Sigma|T|^2$ . For the ratio of the incoherent  $(\mu^-, e^-)$  rate divided by the total muon capture rate, since  $\bar{\Sigma}\Sigma|T|^2$  for the first process is proportional to  $E_\mu E_e$  while for the ordinary muon capture it is proportional to  $E_\mu E_\nu$ , we get

$$R = \frac{\Gamma_{inc}}{\Gamma_{\mu c}} = \frac{m_p m_e [(E_\mu E_e)^{-1} \bar{\Sigma}\Sigma|T|^2 (\mu^- p \rightarrow e^- p)]_{av}}{m_n m_\nu [(E_\mu E_\nu)^{-1} \bar{\Sigma}\Sigma|T|^2 (\mu^- p \rightarrow n \nu_\mu)]_{av}} G_p(Z, N) + \frac{m_n m_e [(E_\mu E_e)^{-1} \bar{\Sigma}\Sigma|T|^2 (\mu^- n \rightarrow e^- n)]_{av}}{m_p m_\nu [(E_\mu E_\nu)^{-1} \bar{\Sigma}\Sigma|T|^2 (\mu^- p \rightarrow n \nu_\mu)]_{av}} \frac{N}{Z} G_n(Z, N) \quad (183)$$

The quantities  $G_p, G_n$  are smooth functions of the momenta and contain the nuclear dependence of the ratio  $R$ . They are defined as

$$G_p(Z, N) = \frac{\int d^3x |\Phi_\mu(\mathbf{x})|^2 (2\pi)^{-3} \int d^3p_e \text{Im} \bar{U}_{p,p}(p_\mu - p_e)}{\int d^3x |\Phi_\mu(\mathbf{x})|^2 (2\pi)^{-3} \int d^3p_\nu \text{Im} \bar{U}_{p,n}(p_\mu - p_\nu)} \quad (184)$$

$$G_n(Z, N) = \frac{Z \int d^3x |\Phi_\mu(\mathbf{x})|^2 (2\pi)^3 \int d^3p_e \text{Im} \bar{U}_{n,n}(p_\mu - p_e)}{N \int d^3x |\Phi_\mu(\mathbf{x})|^2 (2\pi)^3 \int d^3p_\nu \text{Im} \bar{U}_{p,n}(p_\mu - p_\nu)} \quad (185)$$

Note that  $R$  differs from the branching ratio  $R_{eN}$ , because  $R$  does not include the coherent part of the  $(\mu^-, e^-)$  process. The calculation of  $R$  in various nuclei is based upon  $G_p$  and  $G_n$ , since the  $|T|^2$  at the elementary level is in most models independent of the nuclear parameters (see ref. [103] for values of  $|T|^2$  in ordinary muon capture). As an example, we give here the ratio  $R$  in the non-photonic case [12] involving the box diagrams of sect. 3.3.3 which is

$$R = \frac{\Gamma_{inc}(\mu^-, e^-)}{\Gamma(\mu^-, \nu)} = \frac{f_1^2 + f_2^2}{2} \left( \frac{\beta_0(3 + \beta)}{2} \right)^2 G(A, Z) \quad (186)$$

All the nuclear information is contained in the function  $G(A, Z)$  defined as

$$\begin{aligned}
G(A, Z) \equiv & G_p(Z, N) \left[ f_V^2 + 3f_A^2 \left( \frac{\xi + \beta}{3 + \beta} \right)^2 \right] \\
& + G_n(Z, N) \frac{N}{Z} \left[ f_V^2 \left( \frac{3 - \beta}{3 + \beta} \right)^2 + 3f_A^2 \left( \frac{\xi - \beta}{3 + \beta} \right)^2 \right] \quad (187)
\end{aligned}$$

We should mention that the quantities  $G_p$ ,  $G_n$  contain Pauli blocking corrections and for the ordinary  $\mu^-$  capture reaction the Q value, which is significant for light nuclei, is also considered in the argument of the Lindhard function in eq. (181). In heavy nuclei the Q value is approximately given in terms of the difference between the neutron and proton Fermi energies and no other correction is needed.

As a summary we would say that, the use of an exact muon wave function for the calculation of the coherent rate, gives more accurate results in the medium and heavy nuclei region. It involves an approximation (local density approximation) for the sum of the non-coherent contributions. It uses neither closure nor explicit summation over all final states. It makes a summation over the continuum of excited states in a local Fermi sea. Its accuracy is tied to the number of states one can excite; the larger the better. The muon mass provides the energy for the excitation of such states. Furthermore, the method is quite simple to apply in actual calculations.

## 5. EVALUATION OF THE NUCLEAR MATRIX ELEMENTS

In this section we will discuss the methods of calculating the nuclear matrix elements needed for the partial and total  $(\mu^-, e^-)$  conversion rates. It is known that the main feature of the (photonic)  $\mu$ -e conversion rates is the strong Z-dependence (see below eq. (188)) accounted for by assuming that all protons can interact independently with the muon and that this interaction is proportional to the muon density in the 1s atomic orbit at the position of the nucleus.

In this case, as we have explained in sect. 4, we can assume that, the probability density for converting a bound muon in the 1s orbit to an electron

of momentum  $p_e$ , is approximately analogous to the muon average probability density over the nucleus,  $\langle \Phi_{1s} \rangle^2$ , and the nuclear matrix elements  $M_{fi}^2$  between the initial and final nuclear states. Thus, in order to find the partial rate  $\Gamma_{i \rightarrow f}$  or the total rate  $\Gamma = \sum_f \Gamma_{i \rightarrow f}$ , one needs to evaluate the contribution of the nuclear matrix elements  $|\langle f | \Omega | i \rangle|^2$  for the states  $|f\rangle$ .

### 5.1. The coherent $\mu - e$ conversion matrix elements

In the approximation in which one factorizes outside the integral of eq. (173) the average value of the muon wave function  $\Phi_\mu(\vec{x})$  (see eq. (167)), the nuclear dependence of the rate in the coherent process, analogous to the matrix element  $M_{gs \rightarrow gs}^2$ , is written in terms of the proton and neutron elastic nuclear form factors,  $F_Z$  and  $F_N$  (see eq. (176), respectively, as

$$M_{gs \rightarrow gs}^2 = \tilde{g}_V^2 (3 + f_V \beta)^2 Z^2 F_Z^2(q^2) \left[ 1 + \frac{3 - f_V \beta}{3 + f_V \beta} \frac{N}{Z} \frac{F_N(q^2)}{F_Z(q^2)} \right]^2 \quad (188)$$

We note that in the photonic case ( $f_V = 1, \beta = 3$ ) the neutron contribution vanishes and the nuclear matrix element becomes  $Z^2 F_Z(q^2)^2$ . We also mention that the nuclear form factors for the coherent  $\mu - e$  process are calculated at  $q \approx m_\mu = .534 fm^{-1}$ .

The nuclear form factors,  $F_Z(q^2)$  and  $F_N(q^2)$ , have been calculated by using various models i.e. Fermi distribution [98, 99] shell model [100], quasi-particle RPA [101]. In the framework of shell model for spherical nuclei it was found that  $F_Z(q^2)$  and  $F_N(q^2)$  can be cast in tractable analytical forms containing fractional occupation probabilities [107], which take into account a significant part of the nucleon-nucleon correlations. Recently [106], the neutron form factors  $F_N(q^2)$  have been also extracted from the analysis of pionic atoms by means of a two-parameter Fermi distribution for heavy and very heavy nuclei and a harmonic oscillator density for light nuclei. The form factor  $F_Z$  can be obtained from the available electron scattering data [102].

In the context of quasi-particle RPA one can calculate the coherent  $\mu - e$  matrix elements of eq. (188), by using as ground state an uncorrelated or

correlated vacuum. In the first case the nuclear form factors,  $F_Z$  and  $F_N$  take the form

$$F_Z(q^2) = \frac{1}{Z} \sum_j (2j+1) \langle j || j_0(qr) || j \rangle \left( V_j^Z \right)^2 \quad (189)$$

$$F_N(q^2) = \frac{1}{N} \sum_j (2j+1) \langle j || j_0(qr) || j \rangle \left( V_j^N \right)^2 \quad (190)$$

The quantities  $V_j^Z$ ,  $V_j^N$  are the amplitudes for the proton, neutron single particle states to be occupied. Their values are determined by solving the known BCS equations iteratively [108], lie between one and zero and differ from those involved in the independent particle shell model (0 or 1). This is due to the consideration of pairing correlations in the RPA ground state, which deforms the Fermi surface of the nucleus, a picture described by the fractional occupation probabilities. The corresponding shell model form factors  $F_Z$  with fractional occupation probabilities, have been determined [107] by fitting to the electron scattering data.

The accurate determination of the RPA ground state is of great importance for the exact calculation of the coherent and the total  $(\mu^-, e^-)$  rate. The *g.s.* wave function provides the *gs*  $\rightarrow$  *gs* transitions, which are the dominant channel of the  $(\mu^-, e^-)$  process and the total rate in the sum rule approach. A usual correction inserted in the ground state is the *g.s.* correlations [109, 110, 111, 112, 113], which can be included in the ground state by defining the correlated QRPA vacuum  $|\tilde{0}\rangle$  in terms of the uncorrelated vacuum  $|0\rangle$ . By using the Thouless theorem the correlated vacuum  $|\tilde{0}\rangle$  can be written as

$$|\tilde{0}\rangle = N_0 e^{\hat{S}^+} |0\rangle \quad (191)$$

where  $\hat{S}^+$  the operator

$$\hat{S}^+ = \frac{1}{2} \sum_{ab,\tau,JM} \frac{1}{2J+1} C_{ab}^{(J,\tau)} A^+(a, JM) A^+(b, \overline{JM}) \quad (192)$$

The operators  $A^+(a, JM)$  denote the two quasi-particle (or pair) creation operators, in the angular momentum coupled representation. The indices  $a$

and  $b$ , denote proton ( $\tau = 1$ ) or neutron ( $\tau = -1$ ) configurations coupled to  $J$ , i.e.  $a \equiv (j_2 \geq j_1)$  (and similarly for  $b$ ), with  $j_i$  running over the single particle states of the chosen model space:  $j_i \equiv (n_i, l_i, j_i)$ . The correlation matrix  $C$  (symmetric matrix) is constructed for each multipole field  $\lambda$  from the  $X$  and  $Y$  matrices i.e. from the RPA amplitudes for forward and backward excitation. A usual approximation for  $C$  is the following [111]

$$C_{ab}^{(\lambda)} = \left( Y^{(\lambda)} [X^{(\lambda)}]^{-1} \right)_{ab} \quad (193)$$

In eq. (191),  $N_0$  is the normalization constant, which by keeping terms of first order in the correlation matrix  $C$  is given by

$$N_0^2 = \left[ 1 + \frac{1}{2} \sum_{ab, \lambda, \tau} \tilde{C}_{ab}^{(\lambda, \tau)} C_{ab}^{(\lambda, \tau)} \right]^{-1} \quad (194)$$

By using as ground state the correlated RPA vacuum of eq. (191), the coherent rate matrix elements take the form

$$\langle \tilde{0} | \hat{T} | \tilde{0} \rangle = N_0^2 \langle 0 | \hat{T} | 0 \rangle \quad (195)$$

which means that the correlated matrix elements are a rescaling of the uncorrelated ones (see a similar expression in sect. 5.2.2 for the total rate matrix elements).

## **5.2. Total $(\mu^-, e^-)$ Conversion branching ratios**

To find the total  $(\mu^-, e^-)$  conversion rate, one need evaluate the matrix elements for both the vector and axial vector operators (see eq. (166) ) and for all the final nuclear states  $|f\rangle$  i.e. the quantities

$$S_\lambda = \sum_f \left( \frac{q_f}{m_\mu} \right)^2 \int \frac{d\hat{\mathbf{q}}_f}{4\pi} |\langle f | \Omega_\lambda | i \rangle|^2, \quad \lambda = V, A. \quad (196)$$

( $\hat{\mathbf{q}}_f$  is the unit vector in the direction of the momentum transfer  $\mathbf{q}_f$ ). Then the total  $(\mu^-, e^-)$  rate matrix elements are given by

$$M_{tot}^2 = S_V + 3S_A \quad (197)$$

For the calculation of  $S_V$  and  $S_A$ , one can use the following general methods:



1) Summing over partial rates:

With this method we construct explicitly the final nuclear states  $|f\rangle$  in the context of a nuclear model e.g. shell model, random phase approximation, etc. The total  $(\mu^-, e^-)$  rate can be obtained by summing the partial rates for all possible excited states in a chosen model space. By using the multipole expansion of the  $(\mu^-, e^-)$  operators eqs. (163) and (164), the total rate matrix elements  $M_{tot}^2$  can be written as

$$M_{tot}^2 = \sum_s (2s+1) f_s^2 \left[ \sum_{f_{exc}} \left( \frac{q_{exc}}{m_\mu} \right)^2 \sum_{l,J} |\langle f_{exc} || \hat{T}^{(l,s)J} || gs \rangle|^2 + \left( \frac{q_{gs}}{m_\mu} \right)^2 \sum_{l,J} |\langle gs || \hat{T}^{(l,s)J} || gs \rangle|^2 \right] \quad (198)$$

(s=0 for the vector operator and s=1 for the axial vector one). The first term in the brackets of eq. (198) contains the contribution coming from all the excited states  $|f_{exc}\rangle$  of the final nucleus (incoherent rate) and the second term contains the contributions coming from the ground state to ground state channel (coherent rate).

2) Closure approximation:

It is well known that for the description of the total strengths in many processes the sum rule techniques are very useful [31, 63, 64, 114, 115]. In such an approach the contribution of each final state  $|f\rangle$  to the total rate is approximately taken into account without constructing this final state explicitly. One assumes a mean excitation energy of the nucleus  $\bar{E} = \langle E_f \rangle - E_{gs}$  and uses closure for the final states  $|f\rangle$  i.e.  $\sum_f |f\rangle \langle f| = 1$ . This approximation requires the evaluation of one and two-body matrix elements involving only the initial (ground) state. This way the explicit calculation of the final states  $|f\rangle$  is avoided.

The mean excitation energy  $\bar{E}$  of the nucleus, involved in the needed matrix elements is defined as [64]

$$\bar{E} = \frac{\sum_f (E_f - E_{gs}) \left( \frac{|\mathbf{q}_f|}{m_\mu} \right)^2 |\langle f | \hat{T}^J | gs \rangle|^2}{\sum_f \left( \frac{|\mathbf{q}_f|}{m_\mu} \right)^2 |\langle f | \hat{T}^J | gs \rangle|^2} \quad (199)$$

The numerator of this definition represents the energy weighted sum rule and the denominator the non-energy weighted sum rule. Though  $\bar{E}$  is defined in analogy with the ordinary muon capture reaction, the value of the “mean excitation energy” in  $(\mu^-, e^-)$  is different from that in  $(\mu^-, \nu_\mu)$  [101], because in the last process the coherent channel doesn’t exist. For the  $(\mu^-, e^-)$  process the coherent channel ( $gs \rightarrow gs$ ) appears only in the denominator of eq. (199) and, since this dominates the total  $\mu - e$  conversion rate, the resulting mean excitation energy  $\bar{E}$  in this process is much smaller than that for the  $(\mu^-, \nu_\mu)$  reaction. Obviously, the mean excitation energy  $\bar{E}$  can be evaluated by constructing explicitly all the possible excited states  $|f\rangle$  in the context of a nuclear model. In ref. [101], for example, the QRPA method has been used for the determination of the mean excitation energy of the  $^{48}\text{Ti}$  nucleus in the process  $\mu^- + ^{48}\text{Ti} \rightarrow e^- + ^{48}\text{Ti}^*$ .

The method of closure approximation proceeds by defining the operator taken as a tensor product from the single-particle operators  $\hat{T}$  of eq. (163) or (164). For a  $0^+$  initial (ground) state the relevant tensor product is

$$\hat{O} = \sum_{k,k'} [\hat{T}^k \times \hat{T}^{k'}]_0^0 \quad (200)$$

The corresponding total matrix elements in a sum-rule approach are written as

$$M_{tot}^2 = \left(\frac{|\mathbf{k}|}{m_\mu}\right)^2 \left[ f_V^2 \langle i | \hat{O}_V | i \rangle + 3f_A^2 \langle i | \hat{O}_A | i \rangle \right] \quad (201)$$

where  $|\mathbf{k}|$  is the average momentum and the operators  $\hat{O}_V$  (vector) and  $\hat{O}_A$  (axial vector), which contain both one-body and two-body pieces, result from the corresponding  $\hat{T}$  operators of eqs. (163), (164). Consequently, one has to evaluate the matrix elements of the operators  $\hat{O}_V$ ,  $\hat{O}_A$  in a given model.

### 3) Nuclear matter mapped into nuclei with local density approximation:

The method has been already discussed in sects. 4.2.1-4.2.2. The obtained results will be presented in sect. 5.3.

**5.2.1. RPA calculations involving the final states explicitly.** In actual shell model calculations it is quite hard to construct the final states explicitly in medium and heavy nuclei. In such cases we can employ the RPA approximation. In the context of the quasi-particle RPA the final states entering the partial rate matrix elements are obtained by acting on the vacuum  $|0\rangle$  with the phonon creation operator [108, 110, 114]

$$Q^+(fJM) = \sum_{a,\tau} \left[ X_a^{(f,J,\tau)} A^+(a, JM) - Y_a^{(f,J,\tau)} A(a, \overline{JM}) \right] \quad (202)$$

(angular momentum coupled representation) i.e.  $|f\rangle = Q^+ |0\rangle$ . The quantities  $X$  and  $Y$  in eq. (202) are the forward and backward scattering amplitudes. The index  $a$ , runs over proton ( $\tau = 1$ ) or neutron ( $\tau = -1$ ) two particle configurations coupled to  $J$ .

The nuclear matrix element involved in the partial rate  $\Gamma_{i \rightarrow f}$  from an initial state  $0^+$  to an excited state  $|f\rangle$  takes the form

$$\langle f || \hat{T}^{(l,S)J} || 0^+ \rangle = \sum_{a,\tau} W_a^J \left[ X_a^{(f,J,\tau)} U_{j_2}^{(\tau)} V_{j_1}^{(\tau)} + (-)^\theta Y_a^{(f,J,\tau)} V_{j_2}^{(\tau)} U_{j_1}^{(\tau)} \right] \quad (203)$$

The phase  $\theta=0, 1$  for the vector, axial vector operator, respectively [116]. The probability amplitudes  $V$  and  $U$  for the single particle states to be occupied and unoccupied, respectively, are determined from the BCS equations and the  $X$  and  $Y$  matrices are provided by solving the QRPA equations. The quantities  $W_a^J \equiv W_{j_2 j_1}^J$  contain the reduced matrix elements of the operator  $\hat{T}$  between the single particle proton or neutron states  $j_1$  and  $j_2$  as

$$W_{j_2 j_1}^J = \frac{\langle j_2 || \hat{T}^J || j_1 \rangle}{2J+1} \quad (204)$$

In *table 3*, the results of  $^{48}\text{Ti}$  are shown for various values of the parameter  $\beta$  in a model space consisting of all the single particle levels included up to  $3\hbar\omega$  (same for protons and neutrons). In the photonic mechanism mechanism the axial vector matrix elements are zero (see eq. (160)).

Table 3. Incoherent  $\mu - e$  conversion matrix elements: vector ( $S_V$ ), for the photonic mechanism ( $\beta = 3$ ) and vector and axial vector ( $S_A$ ), for a non-photonic mechanism ( $\beta = 5/6$ ). They are for all the excited states included in the up to  $3\hbar\omega$  model space for  $^{48}\text{Ti}$ .

Mode	photonic mechanism	non-photonic mechanism	
$J^\pi$	$S_V$	$S_V$	$S_A$
$0^+$	1.111	2.363	0.0
$1^+$	0.0	0.0	0.297
$2^+$	0.309	0.422	0.046
$3^+$	0.0	0.0	0.050
$4^+$	0.002	0.002	$2 \cdot 10^{-4}$
$5^+$	0.0	0.0	$2 \cdot 10^{-6}$
$6^+$	$2 \cdot 10^{-6}$	$2 \cdot 10^{-6}$	$2 \cdot 10^{-7}$
$0^-$	0.0	0.0	0.818
$1^-$	9.744	17.853	0.795
$2^-$	0.0	0.0	0.670
$3^-$	0.052	0.068	0.011
$4^-$	0.0	0.0	0.010
$5^-$	$8 \cdot 10^{-5}$	$1 \cdot 10^{-4}$	$1 \cdot 10^{-5}$
$6^-$	0.0	0.0	$1 \cdot 10^{-5}$
Total	11.217	20.708	2.697

**5.2.2. Sum-Rules in the context of QRPA.** The RPA sum-rules for the total  $(\mu^-, e^-)$  rate, in the case when an uncorrelated ground state vacuum  $|i\rangle = |0\rangle$  in eq. (201) is used, can be easily obtained. Then the matrix elements  $\langle 0 | \hat{O} | 0 \rangle$  are given by

$$\langle 0 | \hat{O} | 0 \rangle \quad (205)$$

$$\begin{aligned}
&= \sum_J \left\{ \left[ \sum_{j,\tau} (2j+1) < j || \hat{T}^J || j > \left( V_j^{(\tau)} \right)^2 \right]^2 \right. \\
&\quad \left. + (2J+1) \sum_{a,\tau} \tilde{p}(aJ, \tau) p(aJ, \tau) \right\}
\end{aligned}$$

where the first term gives the one-body contribution and the second the two-body contribution. The quantities  $p$ ,  $\tilde{p}$  are given by

$$p(\alpha J, \tau) = W_{j_2 j_1}^J \left[ U_{j_2}^{(\tau)} V_{j_1}^{(\tau)} + (-)^\theta U_{j_1}^{(\tau)} V_{j_2}^{(\tau)} \right] \quad (206)$$

$$\tilde{p}(\alpha J, \tau) = W_{j_2 j_1}^J \left[ V_{j_2}^{(\tau)} U_{j_1}^{(\tau)} + (-)^\theta V_{j_1}^{(\tau)} U_{j_2}^{(\tau)} \right] \quad (207)$$

$a \equiv (j_2 \geq j_1)$ . The phase  $\theta$  is the same as in eq. (203).

The correlated quasi-particle RPA sum-rule of the  $(\mu^-, e^-)$  reaction, which is the expectation value of the operator  $\hat{O}$  of eq. (200) in the correlated ground state  $|\tilde{0}\rangle$  of eq. (191), is written as

$$\langle \tilde{0} | \hat{O} | \tilde{0} \rangle = N_0^2 \left( \langle 0 | \hat{O} | 0 \rangle + \langle 0 | \{ [\hat{S}, \hat{O}] + [\hat{O}, \hat{S}^+] \} | 0 \rangle \right) \quad (208)$$

where the second term in the brackets takes into account the 2p2h excitations. Thus, in order to compute the total rate with the correlated matrix elements,  $\langle \tilde{0} | \hat{O} | \tilde{0} \rangle$ , one should first calculate the uncorrelated matrix element  $\langle 0 | \hat{O} | 0 \rangle$  and afterwards the contribution coming from the ground state RPA correlations. The latter contribution can be cast in the form

$$\begin{aligned}
&\langle 0 | \{ [\hat{S}, \hat{O}] + [\hat{O}, \hat{S}^+] \} | 0 \rangle \\
&= \sum_{ab, \lambda, \tau} C_{ab}^{(\lambda, \tau)} \frac{W_{j_2 j_1}^\lambda W_{j'_2 j'_1}^\lambda}{(1 + \delta_{j_2 j_1})(1 + \delta_{j'_2 j'_1})} \\
&\quad \times \left[ U_{j_2}^{(\tau)} V_{j_1}^{(\tau)} U_{j'_2}^{(\tau)} V_{j'_1}^{(\tau)} \right. \\
&\quad \left. + V_{j_2}^{(\tau)} U_{j_1}^{(\tau)} V_{j'_2}^{(\tau)} U_{j'_1}^{(\tau)} \right]
\end{aligned} \quad (209)$$

The summation (210) gives the proton-proton (p-p) and neutron-neutron (n-

n) correlations. The results obtained for  $^{48}\text{Ti}$  (see *tables 4* and *5*) show that the contribution from the two-body correlations as expected are small.

*Table 4. Total rate nuclear matrix elements and  $gs \rightarrow gs$  transition for the photonic mechanism ( $\mu^-$ ,  $e^-$ ) conversion rates in  $^{48}\text{Ti}$  calculated: 1) with shell model sum-rule and 2) with various types of QRPA sum-rules for different mean excitation energies.*

<i>Method</i>	$M_{gs \rightarrow gs}^2$	$\bar{E}$	$M_{tot}^2$	$\eta$ (%)
<i>Shell Model(sum - rule)</i>	144.6	20.0	188.8	67.2
<i>QRPA(explicit)</i>	135.0	-	161.0	83.9
<i>QRPA(sum - rule)</i>	135.0	1.7	138.3	97.6
<i>QRPA(sum - rule)</i>	135.0	5.0	140.6	96.0
<i>QRPA(sum - rule)</i>	135.0	20.0	141.7	95.3
<i>QRPA + Corr (sum - rule)</i>	87.8	1.7	90.4	97.1
<i>QRPA + Corr (sum - rule)</i>	87.8	5.0	91.8	95.6
<i>QRPA + Corr (sum - rule)</i>	87.8	20.0	92.6	94.8

*Table 5. Total rate matrix elements and  $gs \rightarrow gs$  transitions for  $^{48}\text{Ti}$  given by various methods for the non-photonic mechanism  $\beta = 5/6$  (see caption of table 4).*

<i>Method</i>	$M_{gs \rightarrow gs}^2$	$\bar{E}$	$M_{tot}^2$	$\eta$ (%)
<i>Shell Model(sum - rule)</i>	374.3	20.0	468.0	80.0
<i>QRPA(explicit)</i>	363.0	-	386.4	93.9
<i>QRPA(sum - rule)</i>	363.0	0.5	366.2	99.1
<i>QRPA(sum - rule)</i>	363.0	5.0	376.5	96.4
<i>QRPA(sum - rule)</i>	363.0	20.0	382.8	94.8
<i>QRPA + Corr(sum - rule)</i>	236.2	0.5	238.6	99.0
<i>QRPA + Corr(sum - rule)</i>	236.2	5.0	245.4	96.3
<i>QRPA + Corr(sum - rule)</i>	236.2	20.0	249.5	94.7

**5.2.3 Sum-rules in the context of shell model.** The total  $\mu$ -e conversion rate matrix elements of eq. (201) can be conveniently evaluated by using shell model closure approximation. To obtain the needed matrix elements we assume that the initial nuclear wave function  $|i\rangle$  is a Slater determinant with closed proton and neutron (sub)shells constructed out of single particle harmonic oscillator wave functions. The tensor product operator  $\hat{O}$  of eq. (200) in this case (no second quantization) is written as [115]

$$\hat{O}_\alpha = \tilde{g}_\alpha^2 \sum_{ij} [A_\alpha + B_\alpha(\tau_{3i} + \tau_{3j}) + C_\alpha \tau_{3i} \tau_{3j}] \Theta_{ij}^\alpha, \quad \alpha = V, A \quad (210)$$

where

$$A_V = 9, \quad B_V = 9, \quad C_V = 9, \quad \tilde{g}_V = \frac{1}{6}, \quad \tilde{g}_A = 0 \quad (211)$$

$$\begin{aligned} A_V &= 9/f_V^2, \quad B_V = (3/f_V)\beta, \quad C_V = \beta^2, & \tilde{g}_V &= \frac{1}{2} \\ A_A &= \xi^2/f_V^2, \quad B_A = (\xi/f_V)\beta, \quad C_A = \beta^2, & \tilde{g}_A &= \frac{1}{2} \end{aligned} \quad (212)$$

for the photonic and non-photonic cases correspondingly. Furthermore,

$$\Theta_{ij}^V = j_0(qr_{ij}), \quad \Theta_{ij}^A = j_0(qr_{ij}) \frac{\vec{\sigma}_i \cdot \vec{\sigma}_j}{3} \quad (213)$$

From eq. (210) we can see that, contrary to the  $(\mu^-, \nu_\mu)$  reaction, a distinct feature of the operator  $\hat{O}$ , which is responsible for the process  $(\mu^-, e^-)$ , is the presence of an isospin independent term in both the vector and axial vector component. It has been shown [115] that this part of the  $\mu - e$  operator, is the dominant one. By using standard shell model techniques [31] the matrix elements  $M_{tot}^2$ , which contain one body and two body terms, can be cast in compact analytical forms as

$$S_\lambda = g_\lambda(A, Z) \left( \frac{|\mathbf{q}|}{m_\mu} \right)^2 \left( 1 - \sum_{\kappa=1}^{N_{max}} \xi_\kappa \alpha^{2\kappa} \right) e^{-\alpha^2/4}, \quad \alpha = \sqrt{2} |\mathbf{q}| b, \quad \lambda = V, A \quad (214)$$

where  $b$  is the harmonic oscillator parameter,  $N_{max}$  is the maximum number of oscillator quanta occupied by the nucleons in the considered nucleus and  $\xi_\kappa$  are appropriate coefficients which may depend on  $A$  and  $Z$ . The functions  $g_\lambda(A, Z)$  describe the total rate if its dependence on the momentum transfer can be neglected. The definitions of the coefficients  $\xi_\lambda$  and the functions  $g_\lambda(A, Z)$ , are given in ref. [31].

The total  $(\mu^-, e^-)$  matrix elements using the shell model sum rules take the form

$$M_\alpha^2 = \tilde{g}_\alpha^2 \left[ Z + \left( \frac{3 - f_V \beta}{3 + f_V \beta} \right)^2 N + A_\alpha S_\alpha^{(0)} + B_\alpha S_\alpha^{(1)} + C_\alpha S_\alpha^{(2)} \right], \quad \alpha = V, A \quad (215)$$

where  $S_\alpha^\kappa$ ,  $\kappa = 0, 1, 2$  correspond to the three operators entering eqs. (210)-(213). In *tables 3* and *4* the results obtained this way in  $^{48}Ti$  for average excitation energy  $\bar{E} = 20 MeV$ , are compared with those obtained by using quasi-particle RPA.

**5.2.4. The  $(\mu^-, e^-)$  conversion in the Primakoff's method.** It is well known that success of the sum rule method hinges upon a good choice of mean excitation energy  $\bar{E}$ . Then it requires only knowledge of the structure of the ground state. Very early Primakoff [63] developed a phenomenological method for the  $(\mu^-, \nu_\mu)$  reaction, which reproduces very well the experimental



data for the total muon capture rate and which does not contain the energy  $\bar{E}$ . It yields the well-known Goulard-Primakoff [64] function (see sect. 3.2.2). which contains three parameters, the strengths of the isoscalar, isovector and isotensor parts of the muon-capture operator which are determined by a fit to the total muon capture reaction data of the hole periodic table.

By exploiting the common components of the  $(\mu^-, \nu_\mu)$  and  $(\mu^-, e^-)$  conversion operators one can construct a phenomenological formula for the  $(\mu^-, e^-)$  process [31]. In this way the isospin dependent total  $\mu - e$  rate can be estimated by using the values for the isoscalar, isovector and isotensor parameters of the Goulard-Primakoff function determined by the ordinary muon capture data. Even though the results of this phenomenological method compare well with those of the shell model closure approximation, the fraction of the total  $(\mu \rightarrow e)$  rate coming from the isospin dependent part of the  $(\mu^-, e^-)$  operator is tiny. Furthermore, as we have stressed in sect. 4, the dominant channel in the  $(\mu^-, e^-)$  process is the coherent one which is not possible in the  $(\mu, \nu_\mu)$  and therefore it cannot be obtained from the Goulard-Primakoff method.

### **5.3. Discussion of nuclear matrix elements**

As it can be seen from table 4, the ground state to ground state transition (coherent contribution) exhausts a large portion of the sum rule given by the parameter  $\eta$ . This is not entirely unexpected, since in this case the contribution of all nucleons is coherent. This is very encouraging since, as we have seen in sect. 2.3, the coherent process is of experimental interest. In order to extract  $\eta$  in addition to the coherent rate we must have a reliable estimate of the total rate. The value of  $\eta$  depends, of course, on the parameters  $\tilde{g}_V, \tilde{g}_A$  and  $\beta$  of the elementary amplitude.

The only nuclear information needed for the coherent mode are the “muon-nuclear” form factors  $\tilde{F}_p(q^2)$  and  $\tilde{F}_n(q^2)$ , (see eq. (175)), which can easily be calculated both in the shell model and in RPA calculations. Admittedly, however, these calculations may not be very precise at the high momentum transfer  $q^2 = m_\mu^2 c^2 \sim 0.25 fm^{-2}$ . If on the other hand the muon wave function is assumed to be constant to be taken out of the radial integrals, the nuclear

matrix element essentially depends on the proton and neutron form factors  $F_N(q^2)$  and  $F_Z(p^2)$  which can be taken from experiment (electron scattering data, pionic atoms etc.).

Using the results of table 6 (Chiang *et al.*, 1993 [99]), we can compute the parameter

$$\xi_{p,n}(A, Z) = \left[ \frac{F_{Z,N}^2(p^2)}{\tilde{F}_{p,n}^2(p^2)} \right] \frac{\alpha^3 m_\mu^3}{\pi} Z_{eff}^4 Z \quad (216)$$

This parameter gives us a measure of the deviation of the effective form factor (involving and the muon) from the usual nuclear form factor. The obtained results are presented in *table 7*.

*Table 7. Values of the parameter  $\xi_{p,n}(A, Z)$  of eq. (216) in the text. The approximation of eq. (167) is not very accurate for heavy nuclei.*

(A,Z)	(12,6)	(32,16)	(40,20)	(48,22)	(90,40)	(208,82)
$\xi_p$	0.979	0.962	0.946	0.990	0.850	0.768
$\xi_n$	0.979	0.956	0.951	0.639	0.560	0.285

We thus see that, for heavy nuclei the muon wave function cannot be taken as a constant. This is especially true for the neutron component. This means that in this region the experimental form factors must be used with caution, even though the effect on the branching ratio may be less pronounced.

All calculations of the coherent rate indicate that, in spite of the earlier expectations (Weinberg and Feinberg, 1959 [97], the coherent branching ratio increases all the way up to the Pb region where it starts decreasing.

For the supersymmetric model discussed in sect. 3.3.3, we present our results in *table 8*.

*Table 8. The nuclear form factors  $F_Z$  and  $F_N$  entering in the quasi -elastic  $(\mu, e)$  conversion (eq. (149)). The quantities  $\gamma_{ph}$  and  $\kappa$  are defined in the text (see eqs. (150), (151)), and are evaluated in the case of supersymmetric models.*

A	Z	$F_Z$	$F_N$	$\kappa$	$\gamma_{ph}$	$R_{eN}/R_{e\gamma}$
4	2	0.865	0.865	1.67	1.56	$1.51 \cdot 10^{-3}$
12	6	0.763	0.763	1.67	3.64	$3.53 \cdot 10^{-3}$
14	6	0.753	0.745	1.88	7.96	$9.36 \cdot 10^{-3}$
16	8	0.736	0.736	1.67	4.52	$4.39 \cdot 10^{-3}$
28	14	0.639	0.639	1.67	5.95	$5.78 \cdot 10^{-3}$
32	16	0.618	0.618	1.67	6.37	$6.19 \cdot 10^{-3}$
40	20	0.582	0.582	1.67	7.05	$6.85 \cdot 10^{-3}$
48	20	0.563	0.515	1.85	16.08	$1.84 \cdot 10^{-2}$
48	22	0.543	0.528	1.77	9.74	$1.18 \cdot 10^{-2}$
60	28	0.489	0.478	1.74	9.24	$9.58 \cdot 10^{-3}$
72	32	0.456	0.435	1.79	11.54	$1.25 \cdot 10^{-2}$
82	32	0.440	0.379	1.89	24.98	$3.00 \cdot 10^{-2}$
88	38	0.412	0.370	1.79	12.98	$1.40 \cdot 10^{-2}$
90	40	0.406	0.367	1.76	11.41	$1.20 \cdot 10^{-2}$
114	50	0.335	0.306	1.77	10.35	$1.10 \cdot 10^{-2}$
132	50	0.315	0.250	1.86	25.80	$3.00 \cdot 10^{-2}$
156	64	0.263	0.207	1.76	11.96	$1.30 \cdot 10^{-3}$
162	70	0.253	0.202	1.70	8.92	$8.94 \cdot 10^{-3}$
168	68	0.249	0.191	1.76	12.47	$1.35 \cdot 10^{-2}$
176	70	0.242	0.181	1.76	13.75	$1.49 \cdot 10^{-2}$
194	82	0.198	0.168	1.77	7.20	$7.69 \cdot 10^{-3}$
208	82	0.189	0.135	1.73	10.42	$1.07 \cdot 10^{-2}$

The total rate can also be calculated in the context of the shell model. In this case it is not possible to construct all final states explicitly in realistic

model spaces. One is thus forced to invoke the closure approximation (see sect. 5.2.3). One, however, has no idea about the proper average excitation energy  $\bar{E}_{exc}$  to use. In the earlier calculations (Kosmas and Vergados, 1990 [31]) the value used was the same with that of the common muon-capture. Using this value of  $\bar{E}_{exc} = 20\text{MeV}$ , the value of  $\eta$  ranged from over 90% in light nuclei to about 30% in heavy nuclei (ibid, Table 4). We have seen, however, in sect. 5.2.2 that, with the proper definition of eq. (199),  $\bar{E}_{exc}$  must be quite a bit lower. We should, therefore, give more credibility to the RPA results discussed above (tables 4 and 5).

The calculation of the total rates is quite a bit harder. It can, however, easily be done in the context of QRPA. In QRPA one can also apply the sum rule techniques with an average energy defined by eq. (199), which in this case can be calculated. The agreement between the two methods is quite good (see tables 4 and 5). It is important to note that the ground state correlations tend to decrease all rates in our example [101] by 35%. The obtained value of  $\eta$  is quite high.

The total  $(\mu, e)$  conversion rate has also been computed by a recent new method which utilizes nuclear matter mapped into nuclei with the local density approximation (see sects. 4.2.1 and 4.2.2). This method has the advantage that both the incoherent rates of  $(\mu, e)$  conversion and ordinary muon capture can be computed in the same way (see expressions for  $G_p(N, Z)$  and  $G_n(N, Z)$  of eqs. (184) and (185) as well as eq. (187). Using the results of this calculation we see from table 6 that the parameter  $\eta$  is in all cases greater than 80% and keeps increasing from light to heavy nuclei. Furthermore, the nuclear matrix elements for the coherent mode and the resulting branching ratios increase all the way to the  $Pb$  region.

## 6. CONCLUDING REMARKS

As we have seen in sect. 3, the most popular scenario for lepton-flavor violation involves intermediate neutrinos at the one-loop levels. We have seen, however, that due to the GIM mechanism we encounter an unfavorable explicit dependence on the neutrino mass. We have seen in sec. 3.2.1 that, for leptonic currents of the same helicity the amplitude for light neutrinos

depends on the square of the neutrino mass while for heavy neutrinos, on the inverse neutrino mass squared (see eqs. 80, 81 83, 84). Thus in this mechanism lepton flavor is unobservable for neutrinos much lighter or much heavier than the W-boson mass. This unfortunately happens to be the case with most currently fashionable models. The explicit dependence on the neutrino mass is somewhat improved (linear for light neutrinos, inversely proportional for heavy neutrinos) in the case of L-R interference in the leptonic sector (see eqs. 83, 84 and 85, 86). However, the situation essentially does not improve due to the presence of the mixings  $U^{(12)}$  or  $U^{(21)}$  which in most fashionable models is negligible.

From the phenomenological point of view, therefore, we can add very little to the discussion of a previous review [12] and we will not elaborate further on this mechanism. We will instead summarize the results obtained using other intermediate particles.

We will begin with an extended higgs sector involving non-exotic particles, i.e. two doublets in the Bjorken - Weinberg mechanism. The calculation of the branching ratio is rather complicated since the two loop contribution becomes dominant. It has recently been done by Chang, Hou and Keung [70], and Barr and Zee [71] only for the  $\mu \rightarrow e\gamma$  process. The predictions of this model depend essentially on four parameters:  $\Delta_{tt}$  (the ttH Yukawa coupling), the combination  $\Delta_{e\mu}\cos\phi_\alpha$ , ( $\Delta_{e\mu}$  is the  $e\mu H$  Yukawa coupling and  $\cos\phi_\alpha$  (the  $W^+W^-H$  cubic coupling relative to that of the standard model), the top quark mass  $m_t$  and the Higgs scalar mass  $m_H$ . The authors take the somewhat optimistic choice of  $\Delta_{tt} \simeq 1$  and  $\Delta_{e\mu}\cos\phi_\alpha = 1$  and plot their obtained branching ratio as a function of  $m_t$  and  $m_H$ . For  $m_H$  very large ( $m_H \gg 1TeV$ ) or small ( $m_H \ll 200GeV$ ) their results are essentially independent of  $m_t$  with branching ratios close to the present experimental limit. These authors, however, present their results as though there is one Higgs Boson by assuming that the contributions from the at least two needed Higgs Scalars do not cancel each other.

Unfortunately, the above elaborate calculation does not provide the form factors  $f_{E0}$  and  $f_{M0}$  which do not contribute to  $\mu \rightarrow e\gamma$  but enter into all the other lepton flavor violating processes which involve virtual photons.

The next extension of the standard model involves singly and/or doubly charged Higgs scalars. Lepton flavor violation can be caused by introducing only the singly charged isosinglet both for  $\mu \rightarrow e\gamma$  and  $\mu \rightarrow 3e$ . The branching ratio unfortunately depends on a big power of the not accurately constrained mass of the isosinglet (the 4th for  $\mu \rightarrow e\gamma$ , the 8th for  $\mu \rightarrow 3e$ ). It also depends on the Yukawa couplings  $\lambda_{\mu\tau}$  and  $\lambda_{e\tau}$  of the isosinglet. Thus, one cannot make accurate predictions. One instead can use the present experimental limit for  $\mu \rightarrow e\gamma$  to derive the constraint [50]

$$M_S \geq 94 GeV (10^2 (\lambda_{\mu\tau} \lambda_{e\tau})^{1/2}) \quad (217)$$

The introduction of the doubly charged Higgs scalars (isotriplet or isosinglet, see sect. 3) can give rise to lepton flavor violation at the tree level for  $\mu \rightarrow 3e$  and muonium-antimuonium oscillations which thus become favorable. In the case of the doubly charged isosinglet  $\chi^{++}$  the amplitude for the  $\mu \rightarrow 3e$  process can be written [12] as

$$\mathcal{M} = \tilde{n}_x \frac{G_F}{\sqrt{2}} \frac{1 - p_{12}}{\sqrt{2}} \bar{u}(p_e)(1 + \gamma_5)u(p_\mu)\bar{u}(p_2)(1 - \gamma_5)u(p_1) \quad (218)$$

which leads to the branching ratio

$$R = \frac{1}{8} |\tilde{n}_x|^2 \quad (219)$$

with

$$\tilde{n}_x = \frac{g_{e\mu} g_{ee}}{g^2} \frac{m_W^2}{m_x^2} \quad (220)$$

Once again the branching ratio depends on the inverse 4th power of the unknown mass  $m_x$  of  $\chi^{++}$ . Assuming that  $g_{e\mu} \simeq g_{ee} \simeq 0.1$  and  $m_x = 10^5 GeV$  we obtain

$$\tilde{n}_x = 1.6 \times 10^{-10}, \quad R \simeq 3.2 \times 10^{-17} \quad (221)$$

which is many orders of magnitude away from the planned experiments. The situation with the doubly charged isosinglet is analogous except that  $\gamma_5 \rightarrow -\gamma_5$  and  $\tilde{n}_x \rightarrow \tilde{n}_\Delta$ . Since, neither the mass nor the couplings of the

isosiglet are determined from other experimental data, we can use the present experimental limit to constrain  $\tilde{n}_\Delta$ . We obtain

$$\tilde{n}_\Delta \leq 3 \times 10^{-6} \quad (222)$$

which leads to the constrain

$$\frac{c_{e\mu}c_{ee}}{M_{\Delta^{++}}^2} \leq 2 \times 10^{-10} GeV^{-2} \quad (223)$$

The amplitude for muonium-antimuonium oscillations is the same with that of eq. (218) except that, the antisymmetrization term  $(1 - p_{12})/\sqrt{2}$  is absent. Thus, for the isotriplet we see that the calculated value of  $\tilde{n}_x$  is much smaller than the experimental limit  $n_x < 0.16$  obtained in sect. 2.4. Similarly, the limit  $\tilde{n}_\Delta \leq 3 \times 10^{-6}$  extracted from the experimental limit of  $\mu \rightarrow 3e$  is much smaller than 0.16. This, of course, indicates that if the doubly charged Higgs scalars exist it is much more likely to observe lepton flavor violation in  $\mu \rightarrow 3e$  rather in  $M - \bar{M}$  oscillations.

We will finally discuss lepton flavor violation in supersymmetric theories (see sects. 3.3.1 and 3.3.3). For  $\mu \rightarrow e\gamma$  the relevant equations are (134) and (135). Assuming that  $m_{3/2} = m_{1/2} \simeq 150 GeV$ , we obtain

$$R_{e\gamma} \simeq 8 \times 10^{-15} \quad (224)$$

Similarly, for the  $\mu \rightarrow 3e$  the branching ratio we obtain

$$R_{3e} \simeq 5 \times 10^{-18} \quad (225)$$

i.e.  $\mu \rightarrow e\gamma$  is favorable in this model.

Let us now discuss  $(\mu, e)$  conversion in supersymmetric theories. In addition to the parameter  $\tilde{\eta}$  of (134), which contains both the effect of renormalization and the mixing angles, we encounter all three functions  $f(x)$ ,  $g(x)$  and  $f_b(x)$  we met in  $\mu \rightarrow 3e$ . For the experimentally interesting coherent contribution the branching ratio is given by eq. (149). By noting that  $q^2 = -m_\mu^2$  and  $m_{\tilde{u}} \sim m_{\tilde{e}} \sim m_{3/2}$  and assuming further that the photino is the lightest particle, one is led to eq. (154). In summary then one can write

$$\frac{R_{eN}}{R_{e\gamma}} = \frac{\alpha}{6\pi} \left( \frac{1}{3} + \frac{3}{4}\kappa \right)^2 \gamma_{ph} \quad (226)$$

All the relevant nuclear physics is contained in the parameters  $\kappa$  and  $\gamma_{ph}$ . These parameters are given in *table 7*. For comparison purposes we mention that for the nucleus  $^{48}\text{Ti}$ , which is of experimental interest, the form factors become  $F_Z = 0.538$ ,  $F_N = 0.506$  (QRPA) and  $F_Z = 0.550$  and  $F_N = 0.520$  (Fermi distribution). For the reader's convenience we also present in *table 7* the ratio  $R_{eN}/R_{e\gamma}$ . From this table we see that the coherent effect of all nucleons, tends to enhance the ratio  $R_{eN}/R_{e\gamma}$  compared to the naive estimates mentioned in the introduction. We see that,  $R_{eN}$  still remains about two orders of magnitude less than  $R_{e\gamma}$ . This, however, can be compensated by the desirable experimental signature of  $(\mu, e)$  conversion mentioned in sec. 2.3.

It is apparent that lepton flavor, unlike strangeness, may be absolutely conserved if the Gods of physics bestowed upon the standard model of electroweak interactions absolute authority. Very few people, however, subscribe to this dogma. Lepton flavor violation follows naturally in most extensions of the standard model. Its observation, though, is not going to come easy. The experimental efforts have reached limits which make further improvements extremely difficult. Worst yet the predictions of currently fashionable models are not encouraging them. So, barring unforeseen developments, such efforts may be classified in the pursuit of nothingness. This, under the difficult conditions of present economies may be catastrophic. But the key lies in “unforeseen circumstances”. Historically, this has been not only the rule but the main buty of science. In any case the theoretical predictions reflect present biases and should not deter the continuation of experimental efforts\*.

$$*\nu\nu\nu \nu\nu \nu \text{ το μηδεν, και αιεν ο κοσμος ο μικρος ο μεγας.} \quad (227)$$



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